Existence of Equilibrium in All-Pay Auctions with Price Externalities*

Olivier Bos†

December 1, 2024

Abstract

This paper investigates all-pay auctions with general price externalities under complete information. I show the existence of a mixed-strategy Nash equilibrium using Schauder's fixed-point theorem, as Brouwer's fixed-point theorem cannot apply due to the infinite-dimensional set of distribution functions. These findings have potential applications for future research on contests and charity auctions.

KEYWORDS: All-pay auction, contest, externalities

JEL CLASSIFICATION: C72, D44, D62

1 Introduction

All-pay auctions are usually studied as either auctions or contests. In this game, all bidders must pay their bids, with the highest bidder winning. Applications include fundraising mechanisms, race competitions, and R&D investments.[1] In a race, competitors value the recognition their participation might bring. Therefore, a loser is better off with an effort closer to the winner's effort and a winner with an effort further to the highest loser's effort. In a charity auction, bidders usually care about the charitable purpose and gain wellbeing from the total amount raised. In both situations, participants benefit from a price externality, either dependent or independent of the winner identity.[2]

A well-known result in this literature is that all-pay auctions are optimal mechanisms for raising money for charity (Goeree et al., 2005, Engers and McManus, 2007). However, this is not

*I am grateful to Youcef Askoura, Lucie Ménager, Nikos Pnevmatikos and Xiangyu Qu for helpful discussions. This research has been conducted as part of the Labex MME-DII (ANR11-LBX-0023-01) project. All errors are mine.

†Université Paris-Saclay, ENS Paris-Saclay, Centre for Economics at Paris-Saclay. E-mail: olivier.bos@ens-paris-saclay.fr.

[1]An investment game is discussed on p.2, where only the winner receives a bid-dependent payoff.

[2]There is also an extensive literature on auctions with allocative externalities initiated by Jehiel and Moldovanu (1996) with complete information. In that context the winner's identity is central, and not, as here, the money/efforts spent by all opponents. For an analysis of all-pay auctions with allocative externalities, see Klose and Kovenock (2015). Following Ettinger (2010), I use the term *price externality* to distinguish concerns about others' payments/efforts from allocative externalities.

consistently supported in practice. In field experiments, Carpenter et al. (2008) and Onderstal et al. (2013) observe low participation, identifying alternative, more effective fundraising mechanisms. This may stem from the assumption of a linear externality that participants benefit fromâa core assumption underlying most theoretical results in this literature.[3] A deeper understanding of non-linear preferences, particularly in the context of all-pay auctions, could yield significant insights into charity auctions. First, it might explain the field-observed puzzle. Second, it could enhance our understanding of the role of all-pay auctions among other commonly used mechanisms, such as lotteries, in charitable settings. Finally, it could guide the design of optimal mechanisms, providing better policy implications. This paper is a step in that direction.

I investigate the all-pay auction with complete information without specifying the form of the externality functions. Complete information is less common in auction theory than in contest literature; however, recent studies on auctions with externalities consider settings with complete information.[4]

The analysis does not assume a specific analytical form for the externality function, allowing bidders to consider both positive externalities from their own bids and either positive or negative externalities from their rivals' bids.[5] The externality function may also capture a relationship between a bidder's bid and her competitors' bids.[6] Such flexibility makes this model particularly relevant for economic applications, especially in charity auctions and race competitions. I establish the existence of a mixed-strategy Nash equilibrium, defined over a closed, convex, infinite-dimensional set of continuous distribution functions. To achieve this, I use Schauder's fixed-point theorem, as the infinite-dimensional nature of the function space precludes the use of Brouwer's fixed-point theorem. This result could be valuable for future research on fundraising mechanisms.

This paper also contributes to the literature on contests with non-monotonic payoffs. Studies in this area examine all-pay auctions with prize values that depend on bids, offering novel insights into R&D races (see, for example, Chowdhury (2017)). Consider an investment game. Firms incur costs to develop innovations that improve either their technology or products before entering a competitive market (e.g., in Bertrand competition). All firms bear these costs as sunk expenses, with the firm investing the most for the best innovation winning the largest market

[3]Among other possible explanations, Carpenter et al. (2010) suggests unfamiliarity with the mechanism and endogenous participation (not considered here), while Bos (2016) notes bidder heterogeneity.

[4]For instance, Jehiel and Moldovanu (1996) examine the impact of allocative externalities on participation in first-price, winner-pay auctions; Konrad (2006) studies ownership structures through an all-pay auction with firm-identity-dependent externalities; Ettinger (2010) investigates first-price and second-price winner-pay auctions with price externalities; Klose and Kovenock (2015) analyze all-pay auctions with allocative externalities; Bos (2016) compares the effectiveness of all-pay versus winner-pay auctions for charity; and Damianov and Peeters (2018) compare the lowest-price all-pay auction with other fundraising mechanisms. Furthermore, Krakel et al. (2014) explore specific externalities in a Tullock contest, yielding consistent findings for all-pay auctions.

[5]Examples include charity auctions, where bidders benefit from the total revenue raised (therefore from all bids including their own), and contests, where a participant may value her own effort positively and her rivals' effort negatively.

[6]This could represent, for instance, a form of correlation.

share, judged relative to its expenses. This analysis differs from mine in two ways. First, in that literature, the bid-dependent payoff applies only to the winner, who does not consider an externality from opponents' bids. As a result, a pure-strategy Nash equilibrium exists in this setting (Amegashie, 2001, Sacco and Schumtzler, 2008, Siegel, 2014, Chowdhury, 2017). By contrast, I establish the existence of a mixed-strategy Nash equilibrium when the bid supported is linearly separable from the prize value.

The remainder of this paper is structured as follows. Section 2 introduces the formal setting and properties of general price externalities. In Section 3, I discuss the non-existence of a pure-strategy Nash equilibrium and demonstrate the existence of a mixed-strategy Nash equilibrium. Section 4 concludes. In the following, I refer to players as *bidders*, using auction terminology.

2 The Model

Suppose n risk-neutral bidders submit their bids for an indivisible object (or prize) allocated to the highest bidder. Let $\mathcal{N} = \{1, ..., n\}$ denote the set of bidders. Bidder i's value is given by v_i, where $v_1 \geqslant v_2 \geqslant ... \geqslant v_n \geqslant 0$. Although valuations are common knowledge among the potential bidders, the seller has no information about them. All bidders must pay their bids. Thus, if bidder i wins with bid x_i, her payoff is $v_i - x_i$; otherwise, she receives $-x_i$.[7] Moreover, the money raised from each potential bidder affects the utility of each other, as each bidder benefits from their own participation and either benefit or suffer from their rivals' bids.

The bidder's utility function thus includes a price externality that depends on all bids. This may take the form of a function of the sum of the bids $\sum_{j=1}^{n} x_j$, as in charity auctions, and thus be independent of the winner's identity. Alternatively, it may depend on the difference between bids contingent on the winner's identity. In contests, participants may get a reward not only from winning but also from her relative performance. Accordingly, I consider an externality function that depends on all bids,

$$h_i : [0, \bar{x}_i] \times_{j=1, j \neq i}^{n} [0, \bar{x}_j] \to \mathbb{R}_+$$

where $\bar{x}_k$ represents the maximum bid of bidder k, for all $k \in \mathcal{N}$. It follows that bidder i's utility is given by[8]

$$U_i(x_i, \boldsymbol{x}_{-i}) = \begin{cases} v_i - x_i + h_i(x_i, \boldsymbol{x}_{-i}) & \text{if } i \text{ is the only winner} \\ \frac{v_i}{k} - x_i + h_i(x_i, \boldsymbol{x}_{-i}) & \text{if } i \text{ is one of the } k \text{ winners} \\ -x_i + h_i(x_i, \boldsymbol{x}_{-i}) & \text{otherwise} \end{cases} \tag{1}$$

with $\boldsymbol{x}_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. In the linear case, this takes the form $\alpha_i \sum_{j=1}^{n} x_j$, where α_i is a positive number. This is the standard representation of externality functions in the charity auction literature[9] and in studies on auctions with identity-independent price externalities.[10] For the analysis, I make the following assumptions, which are relevant for economic applications:

[7] If multiple bidders submit the same winning bid, each wins with equal probability.

[8] Bidder i may be one of k bidders submitting the same highest bid, with $k = \#\{j | j = \arg\max\{x_i, i \in \mathcal{N}\}\}$.

[9] See Goeree et al. (2005), Engers and McManus (2007), and Bos (2016).

[10] See, for example, Ettinger (2010).

Remark that if the externality function were independent of opponents' bids, i.e., $h_i(x_i, \boldsymbol{x}_{-i}) = h_i(x_i)$, the utility function would correspond to a special case of bid-dependent prizes.[11] Additionally, Assumption $A3$ precludes bidders from experiencing a negative externality from their own bids, as explored in Sacco and Schumtzler (2008).

Assumption 1 ($A1$). *If all bidders submit zero bids, none derive any benefit from the externality:* $h_i(0, \ldots, 0) = 0$.

Assumption 2 ($A2$). *The externality function is a* $\mathcal{C}^1$ *function.*

Assumption $A1$ ensures that non-participation by all bidders does not generate any benefit. Assumption $A2$ rules out discontinuities, which could arise in specific environments.[12]

Assumption 3 ($A3$). *Bidder i benefits from a weakly increasing externality in her own bid over* $[0, \bar{x}_i]$.

This means that $\frac{\partial h_i}{\partial x_i}(x_i, \boldsymbol{x}_{-i}) \geqslant 0$. However, the effect of rivals' bids on bidder i's utility depends on the specific form of h_i and how she perceives her competitors' bids and the link between all bids.

Assumption 4 ($A4$). *Bidder i's payoff decreases weakly with her payment $x_i \in [0, \bar{x}_i]$ in the auction.*

Assumption $A4$ means that bidders always prefer lower payments, regardless of whether they win or lose, leading to $\frac{\partial h_i}{\partial x_i}(x_i, \boldsymbol{x}_{-i}) \leqslant 1$. In the literature on all-pay auctions with price externalities, it is commonly assumed, explicitly or implicitly, that payoffs cannot increase with the bidder payment. This implies $\frac{\partial h_i}{\partial x_i}(x_i, \boldsymbol{x}_{-i}) < 1$. In the context of charity auctions, this assumption highlights the limits of altruism; otherwise, bidders might become indifferent between donating and retaining money for personal use.[13] In contests, such as races, competitors' payoffs may depend on both absolute and relative performances. For example, a competitor may gain a higher payoff from a better relative performance, $\alpha_i(x_i - \max_{j \neq i} x_j)$, due to reputation or recognition. However, the marginal cost of exerting effort must exceed the marginal benefit to avoid infinite effort. This condition requires $\alpha_i < 1$, equivalent here to $\frac{\partial h_i}{\partial x_i}(x_i, \boldsymbol{x}_{-i}) < 1$. The investigation here is more general. It also consider when bidders in charity auctions could be fully altruistic or where marginal costs equal marginal benefits in contests. The following example illustrates a plausible non-linear externality function h.

Example 1 (Non-linear externality). *In a contest, such as a race, the perception of a competitor may depend on her relative performance and the absolute performance of all participants. Suppose the winner's relative effort compared to the highest loser's effort is 0.25. She gains a better reputation if her absolute effort exceeds 1, but a worse reputation if it is only 0.5. For the highest loser, being close to the top effort is rewarded for smaller absolute efforts. This means*

[11] See, for instance, the utility forms in Amegashie (2001) and Bos and Ranger (2018).

[12] For example, discontinuities may occur if bidders care about their rank among bids.

[13] For instance, Goeree et al. (2005) and Bos (2016) consider the linear case $\alpha \sum_{i=1}^{n} x_i$ and assume $\alpha < 1$. Engers and McManus (2007) introduce a *warm glow* effect à la Andreoni (1989), where bidders derive different utilities from their own and others' bids, modeled as $\alpha x_i + \beta \sum_{j \neq i}^{n} x_j$, with $1 > \alpha \geqslant \beta > 0$.

that the strength of the relative effort plays a different role for the winner and the losers. These effects can be represented by the function:

$$\alpha_i \left(x_i - (\max_{j \neq i} x_j)^{\frac{1}{2}} \right), \quad \textit{with } \alpha_i < 1, \ \forall i, j \in \mathcal{N}, \ i \neq j.$$

Note that this function satisfies assumptions A1 through A4.

3 Existence of an Equilibrium

In this section, I show the existence of a Nash equilibrium in mixed strategies. To better understand the implications of this result, I first discuss how the (non)existence of a Nash equilibrium in pure strategies depends on the shape of the externality functions.

It is a well-known result that there is no bidding equilibrium in pure strategies in all-pay auctions without price externalities (see Hillman and Riley (1989) and Baye et al. (1996)). Unsurprisingly, this also holds for the class of externality functions where $\frac{\partial h_i}{\partial x_i}(x_i, \boldsymbol{x}_{-i}) < 1$ for all $i = 1, \dots, n$. A sketch of this proof is provided in the Appendix.

Thus, the existence of a bidding equilibrium in pure strategies is not guaranteed and fundamentally depends on a specific and favorable structure of the externality functions. Consequently, I focus on establishing the existence of a Nash equilibrium in mixed strategies. In the following, I denote $F_i(x) \equiv \mathbb{P}(X_i \leqslant x)$ as the cumulative distribution function, representing the probability that bidder i submits a bid lower than x. The functions $F_1, \dots, F_n$ can be interpreted as the bidding (mixed) strategies, where the support is a strict subset of $\mathbb{R}_+$. Regardless of the auction outcome, when bidder i computes her expected utility, she accounts for the bids submitted by all participants, including her own. Notably, we do not know whether the cumulative distributions $F_1, \dots, F_n$ admit density functions. For this reason, I use the Stieltjes integral to calculate the expected utility and to determine the existence of a mixed strategy Nash equilibrium.[14] Furthermore, the Stieltjes integral exists if the cumulative distribution F_i is a function of bounded variation and then is discontinuous in at most a countable set of points.[15] Consequently, these cumulative distributions may lack density functions but can include atoms and a finite number of discontinuities, as in the standard case with no externalities (Baye et al., 1996).

The following lemma establishes that a result by Siegel (2009), regarding atoms in general contests without externalities, also applies to all-pay auctions with general price externalities.

Lemma 1. *Assume that at least two bidders have an atom at x. Then all bidders with an atom at x lose with probability 1 by submitting x.*

Proof. Consider the set $\mathcal{T}$, with $|\mathcal{T}| \geqslant 2$, representing the bidders who have an atom at x, meaning they bid x with strictly positive probability. I will show that the event *a winning*

[14] See, for example, Carothers (2000, Chap. 14) for details on using the Stieltjes integral in cases where density functions do not exist.

[15] A cumulative distribution can be constructed as the difference between two bounded monotone functions, making it a function of bounded variation.

tie occurs at x has zero probability. Assume, to the contrary, that this has strictly positive probability. In this case, the object is allocated among the $|\mathcal{T}|$ winners, such that each $i \in \mathcal{T}$ receives her value divided by the number of winners, i.e., $\frac{v_i}{|\mathcal{T}|}$. For bidder i, submitting x with positive probability must lead to a positive payoff:

$$\frac{v_i}{|\mathcal{T}|} - x + h_i(x, \tilde{\boldsymbol{x}}_{-i}) \geqslant 0,$$

for a given vector of other bids $\tilde{\boldsymbol{x}}_{-i}$. Then, there exists an $\epsilon > 0$ such that:

$$v_i - (x + \epsilon) + h_i(x + \epsilon, \tilde{\boldsymbol{x}}_{-i}) \geqslant \frac{v_i}{|\mathcal{T}|} - x + h_i(x, \tilde{\boldsymbol{x}}_{-i}),$$

i.e., for which bidder i can increase her probability of winning to 1 and improve her payoff by submitting $x + \epsilon$. Therefore, *a winning tie occurs at* x must have zero probability. This implies that at least one bidder in $\mathcal{N} \backslash \mathcal{T}$ submits a higher bid than x and wins the auction. Hence, all bidders with an atom at x lose with probability 1 by submitting x.

■

Lemma 1 implies that ties, where bidders win with a strictly positive probability, are zero probability events. As a result, bidder i's expected utility is given by,[16]

$$\mathbb{E}U_i(x_i, \boldsymbol{X}_{-i}) = \Pi_{j \neq i} F_j(x_i) v_i - x_i + \mathbb{E}h_i(x_i, \boldsymbol{X}_{-i}),$$

for all $i = 1, \ldots, n$, where $\boldsymbol{X}_{-i} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$. A potential bidder participates in the auction if there exists a positive bid for which her expected utility is at least equal to the externality she benefits by bidding zero. Formally:

$$\exists x_i \text{ such that } \mathbb{E}U_i(x_i, \boldsymbol{X}_{-i}) \geqslant \mathbb{E}\left(h_i(0, \boldsymbol{X}_{-i})\right)$$

If the externality is linear, a closed-form solution is straightforward to determine. In this case, the expected payoff without externalities is merely subject to an affine transformation. Since the result in Baye et al. (1996) is invariant to affine transformations of expected utility, the mixed strategies are also invariant. Unfortunately, determining an analytic solution is not feasible without providing a specific shape of the externality function h_i. This reflects the general mapping between x_i and $\boldsymbol{x}_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Nevertheless, I am able to establish the existence of a bidding equilibrium in mixed strategies.

A substantial step in determining the existence of a Nash equilibrium was achieved by Reny (1999) through the better-reply security approach: any compact, bounded, and measurable game with a strategy space that is a subset of a Hausdorff linear topological space has a Nash equilibrium if its mixed extension satisfies the better-reply security condition. However, to determine

[16] Remark that $\mathbb{E}\left(h_i(x_i, \boldsymbol{X}_{-i}) \mid \max_{j \neq i} X_j \leqslant x_i\right) = \begin{cases} \dfrac{1}{\Pi_{j \neq i} F_j(x_i)} \displaystyle\int_{[0, x_i]^n} h_i(x_i, \boldsymbol{x}_{-i}) \Pi_{j \neq i} dF_j(x_i) \text{ if } x_i > 0 \\ 0 \text{ otherwise} \end{cases}$ and $\mathbb{E}\left(h_i(x_i, \boldsymbol{X}_{-i}) \mid \exists j \neq i, X_j > x_i\right) = \begin{cases} \dfrac{1}{1 - \Pi_{j \neq i} F_j(x_i)} \displaystyle\int_{([0, x_i]^n)^c} h_i(x_i, \boldsymbol{x}_{-i}) \Pi_{j \neq i} dF_j(x_i) \text{ if } x_i > 0 \\ 0 \text{ otherwise} \end{cases}$ with $([0, x_i]^n)^c$ the complement of $[0, x_i]^n$.

the payoff security of the mixed extension can be quite challenging, and recent studies propose alternative conditions that are easier to assess.[17] For instance, Prokopovych and Yannelis (2014) introduced the uniform diagonal security condition, and Monteiro and Page Jr (2007) developed the uniform payoff security condition. While both conditions ensure the existence of a Nash equilibrium with mixed strategies in standard all-pay auctions, they are not applicable to all-pay auctions with price externalities. For example, consider a case where $x_i = 0$ and $x_j \neq 0$ for all $j \neq i$, in this scenario, the uniform payoff security condition is not satisfied.

Alternatively, Simon and Zame (1990) establishted other existence results for discontinuous games, which could be applied here. Following their results, showing the existence of a Nash equilibrium with mixed strategies would require to determine a new game with an *endogenous* tie-breaking rule. However, this approach would entail substantially more complex work compared to the approach considered in this paper, which use ordered fixed-point theorem.

Moreover remark that the results of Becker and Damianov (2006), which rely on the Glicksberg and Fan fixed-point theorems, cannot be applied here due to the heterogeneity of values.

Proposition 1. *Given assumptions A1 − A4, a mixed strategy Nash equilibrium exists.*

The proof leverages Schauder's fixed-point theorem to establish the existence of the equilibrium. The solution is defined on a closed and convex set of continuous distribution functions, which is a subset of a Hausdorff linear topological space. This approach is necessary because the infinite-dimensional set of distribution functions precludes the use of fixed-point theorems like Knaster-Tarski's or Brouwer's. Other applications of the Schauder's fixed-point theorem are provided by Stokey et al. (1989) for overlapping-generations models, Fudenberg and Tirole (1991) for mixed strategy Nash equilibria with uncountable actions sets, Amir (1996), Curtat (1996) and Balbus et al. (2015) for Markov equilibrium in stochastic intergenerational games, Anderson et al. (1998) for logit equilibria in all-pay auctions and Fey (2008) for pure strategy Bayesian-Nash equilibria in rent-seeking contests.

Proof of Proposition 1. As in Anderson et al. (1998), the proof of Schauder's fixed-point theorem consists in two steps[18]: first, by defining a continuous mapping, and second, by identifying a set of functions that are uniformly bounded and equicontinuous.

In a preliminary step toward determining a fixed point of this mapping, I first examine the indifference principle in my setting, leading to equation 2. This, in turn, establishes a characterization of the mapping in equation 3.

Preliminary Steps.

Let us consider two bidders, i and j. Note that if bidder i has the highest maximum bid, $\bar{x}_i$, they receive a payoff of $v_i - \bar{x}_i + h_i(\bar{x}_i, x_j)$ with probability 1. Given Assumption A4, where $\frac{\partial h_i}{\partial x_i}(x_i, x_j) \leqslant 1$, a reduction in $\bar{x}_i$ to the maximum bid of her rival, $\bar{x}_j$, increases her payoff to $v_i - \bar{x}_j + h_i(\bar{x}_j, x_j)$ without affecting her winning probability. Following the standard reasoning

[17] See Reny (2020) for a discussion.

[18] However, I address a completely different problem than Anderson et al. (1998), who establish the existence of a logit equilibrium in all-pay auctions.

for multiple bidders developed by Baye et al. (1996), all active bidders must have the same maximum bid, $\bar{x}$.[19]

Preliminary Step 1.

I now investigate the indifference principle, which asserts that if all opponents of bidder i adopt mixed strategies, bidder i is indifferent among all pure strategies in the possible best-response set. In other words, bidder i's expected utility is constant for any pure strategy profile of their opponents with positive probability. Let $G_i := \prod_{j \neq i} F_j$, and let the vector $\boldsymbol{y}_{-i} = (y_1, ..., y_{i-1}, y_{i+1}, ..., y_n)$. Given Assumption A2 and using the Stieltjes integral, the indifference principle leads to the derivative:

$$G_i'(x) = \frac{1}{v_i} - \frac{1}{v_i} \int_{[0,\bar{x}]^{n-1}} \frac{\partial h_i}{\partial x}(x, \boldsymbol{y}_{-i}) \Pi_{j \neq i} dF_j(y_j) \text{ for all } i = 1, ..., n. \tag{2}$$

Preliminary Step 2.

Next, I establish that equation (2) is well-defined. Note that the right-hand side of equation (2) requires G_i to be a continuous distribution function. Define $\boldsymbol{G}(x) = (G_1(x), ..., G_n(x))$ as the vector of mixed strategies, and let $\mathcal{D}_i = \{G_i \in \mathcal{C}([0, \bar{x}]) : \|G_i\| \leqslant 1\}$, with $\| \cdot \|$ the supremum norm and $\mathcal{C}([0, \bar{x}])$ the set of continuous functions on $[0, \bar{x}]$. It follows the set $\boldsymbol{\mathcal{D}} \equiv \mathcal{D}_1 \times \cdots \times \mathcal{D}_n$ is equipped with the norm $\|\boldsymbol{G}\|_\infty = \max_{i=1,...,n} \|G_i\|$. Define the mapping $T : \boldsymbol{\mathcal{D}} \to \boldsymbol{\mathcal{D}}$ such that $\boldsymbol{G}(x) \mapsto \boldsymbol{TG}(x) = (TG_1(x), ..., TG_n(x))$. Then, T maps every $G_i \in \mathcal{D}_i$ to $\mathcal{D}_i$, and by integrating equation (2), it is characterized by

$$TG_i(x) \equiv \lambda_i x - \lambda_i \int_{[0,\bar{x}]^{n-1}} h_i(x, \boldsymbol{y}_{-i}) \Pi_{j \neq i} dF_j(y_j) \text{ for all } i = 1, ..., n. \tag{3}$$

with $\lambda_i = \frac{1}{v_i}$.

Remark that the set $\boldsymbol{\mathcal{D}}$, which includes all continuous distribution functions, is closed, convex, and infinite-dimensional, forming a subset of a Hausdorff linear topological space.[20] Schauder's fixed-point theorem is then employed to prove that $\boldsymbol{G}(x)$ is a fixed point of the operator T defined by equation (3). It provides a sufficient condition for mixed strategies to constitute a fixed point of T, ensuring no profitable deviations exist outside the support $[0, \bar{x}]$.

Main Steps.

Theorem 1 (Schauder, 1930). *If $\boldsymbol{\mathcal{D}}$ is a closed convex subset of a normed space and $\boldsymbol{\mathcal{E}}$ a relatively compact subset of $\boldsymbol{\mathcal{D}}$, then every continuous mapping of $\boldsymbol{\mathcal{D}}$ to $\boldsymbol{\mathcal{E}}$ has a fixed-point.*

[19] The proof for n bidders and all possible rankings of valuations is analogous to the one without externalities in Baye et al. (1996).

[20] Equation (2) guarantees the distribution functions G_i are continuous.

Therefore, I show in the following that $\boldsymbol{\mathcal{E}} \equiv \{\boldsymbol{TG} \mid \boldsymbol{G} \in \boldsymbol{\mathcal{D}}\}$ is relatively compact and that T is a continuous operator from $\boldsymbol{\mathcal{D}}$ to $\boldsymbol{\mathcal{E}}$.

Step 1. *$\boldsymbol{\mathcal{E}}$ is relatively compact.*

I use the Arzelà-Ascoli theorem to characterize relative compactness in the space of continuous functions $\mathcal{C}([0,\bar{x}])$.

Theorem 2 (Arzelà-Ascoli, 1895)**.** *A set of functions in $\mathcal{C}([0,\bar{x}])$ with the supremum norm is relatively compact if and only if it is uniformly bounded and equicontinuous on $[0,\bar{x}]$.*

Thus, to establish that $\boldsymbol{\mathcal{E}} \equiv \{\boldsymbol{TG} \mid \boldsymbol{G} \in \boldsymbol{\mathcal{D}}\}$ is relatively compact, I prove that $\boldsymbol{\mathcal{E}}$ is uniformly bounded and equicontinuous on $[0,\bar{x}]$.

First let us show that $\boldsymbol{\mathcal{E}}$ is uniformly bounded. Assumption $A4$ implies that

$$\int_{[0,\bar{x}]^{n-1}} \frac{\partial h_i}{\partial x}(x,\boldsymbol{y}_{-i})\Pi_{j\neq i} dF_j(y_j) \leqslant 1.$$

$TG_i(x)$ is thus increasing and $|TG_i(x)| \leqslant TG_i(\bar{x}) = 1$, for all $x \in [0,\bar{x}]$, $G_i \in \mathcal{D}_i$, $i = 1,...,n$. Therefore, $\boldsymbol{TG}$ is uniformly bounded for all $\boldsymbol{G} \in \boldsymbol{\mathcal{D}}$.

Next, let us prove that $\boldsymbol{TG}$ is equicontinuous $\forall \boldsymbol{G} \in \boldsymbol{\mathcal{D}}$: $\forall \varepsilon, \exists \eta : |TG_i(x_1) - TG_i(x_2)| < \varepsilon$ when $|x_1 - x_2| < \eta$, $\forall G_i \in \mathcal{D}_i$ and $i = 1,...,n$. To show this, notice that the function h_i is continuous and bounded on the compact $[0,\bar{x}]$. I can then compute,

$$\begin{aligned}
|TG_i(x_1) - TG_i(x_2)| &= \left|\lambda_i(x_1 - x_2) - \lambda_i \int_{[0,\bar{x}]^{n-1}} [h_i(x_1,\boldsymbol{y}_{-i}) - h_i(x_2,\boldsymbol{y}_{-i})]\Pi_{j\neq i} dF_j(y_j)\right| \\
&\leqslant \lambda_i\left[|x_1 - x_2| + \left|\int_{[0,\bar{x}]^{n-1}} [h_i(x_1,\boldsymbol{y}_{-i}) - h_i(x_2,\boldsymbol{y}_{-i})]\Pi_{j\neq i} dF_j(y_j)\right|\right] \\
&\leqslant \lambda_i|x_1 - x_2|\left[1 + \frac{|\sup_{\boldsymbol{y}_{-i}\in[0,\bar{x}]^{n-1}}[h_i(x_1,\boldsymbol{y}_{-i}) - h_i(x_2,\boldsymbol{y}_{-i})]|}{|x_1 - x_2|}\right] \\
&< \lambda_i\eta\left[1 + \frac{|\sup_{\boldsymbol{y}_{-i}\in[0,\bar{x}]^{n-1}}[h_i(x_1,\boldsymbol{y}_{-i}) - h_i(x_2,\boldsymbol{y}_{-i})]|}{|x_1 - x_2|}\right].
\end{aligned}$$

Thus, $|TG_i(x_1) - TG_i(x_2)| < \varepsilon$ for $\eta = \varepsilon \min\limits_{i=1,...,n} \dfrac{|x_1 - x_2|}{\lambda_i(|x_1 - x_2| + \kappa_i)}$ for all $G_i \in \mathcal{D}_i$ and $i = 1,...,n$, with $\kappa_i \equiv |\sup_{\boldsymbol{y}_{-i}\in[0,\bar{x}]^{n-1}}[h_i(x_1,\boldsymbol{y}_{-i}) - h_i(x_2,\boldsymbol{y}_{-i})]|$.

Step 2. *T is a continuous mapping from $\boldsymbol{\mathcal{D}}$ to $\boldsymbol{\mathcal{E}}$.*

To establish T is a continuous mapping, I define $\hat{G}_i(x) = \Pi_{j\neq i}\hat{F}_j(x)$ and $\tilde{G}_i(x) = \Pi_{j\neq i}\tilde{F}_j(x)$ such as $\hat{G}_i(x) = \tilde{G}_i(x) + k_i(x)$ with $|k_i(x)| < \eta$ for all $x \in [0,\bar{x}]$, $i = 1,...,n$, and show that for all $\hat{\boldsymbol{G}}, \tilde{\boldsymbol{G}} \in \boldsymbol{\mathcal{D}}$ and for all $\varepsilon > 0$, there is a $\eta > 0$ such that $||\boldsymbol{T}\hat{\boldsymbol{G}}(x) - \boldsymbol{T}\tilde{\boldsymbol{G}}(x)||_\infty < \varepsilon$ when $||\hat{\boldsymbol{G}} - \tilde{\boldsymbol{G}}||_\infty < \eta$. I can compute,

$$
\begin{aligned}
|T\hat{G}_i(x) - T\tilde{G}_i(x)| &= \left| -\lambda_i \left(\int_{[0,\bar{x}]^{n-1}} h_i(x, \boldsymbol{y}_{-i}) \Pi_{j\neq i} d\hat{F}_j(y_j) - \int_{[0,\bar{x}]^{n-1}} h_i(x, \boldsymbol{y}_{-i}) \Pi_{j\neq i} d\tilde{F}_j(y_j) \right) \right| \\
&\leqslant \lambda_i \sup_{\boldsymbol{y}_{-i}\in[0,\bar{x}]^{n-1}} h_i(\bar{x}, \boldsymbol{y}_{-i}) \left| \int_{[0,\bar{x}]^{n-1}} \Pi_{j\neq i} d\hat{F}_j(y_j) - \int_{[0,\bar{x}]^{n-1}} \Pi_{j\neq i} d\tilde{F}_j(y_j) \right| \\
&= \lambda_i \sup_{\boldsymbol{y}_{-i}\in[0,\bar{x}]^{n-1}} h_i(\bar{x}, \boldsymbol{y}_{-i}) |k_i(\bar{x})| \\
&< \lambda_i \sup_{\boldsymbol{y}_{-i}\in[0,\bar{x}]^{n-1}} h_i(\bar{x}, \boldsymbol{y}_{-i}) \eta.
\end{aligned}
$$

Since h_i is a continuous function in all arguments, bounded by $\sup_{\boldsymbol{y}_{-i}\in[0,\bar{x}]^{n-1}} h_i(\bar{x}, \boldsymbol{y}_{-i})$, the second line follows. The transition from the second to the third line comes from the independence of the density functions and $\hat{G}_i(x) - \tilde{G}_i(x) = k_i(x)$. Therefore, $||\boldsymbol{T}\hat{\boldsymbol{G}}(x) - \boldsymbol{T}\tilde{\boldsymbol{G}}(x)||_\infty$ is inferior to $\varepsilon > 0$ when $\eta = \min\limits_{i=1,\ldots,n} \dfrac{\varepsilon}{\lambda_i \sup_{\boldsymbol{y}_{-i}\in[0,\bar{x}]^{n-1}} h_i(\bar{x}, \boldsymbol{y}_{-i}))}$ for all $x \in [0, \bar{x}]$.

■

4 Concluding Remarks

I established the existence of a mixed strategy Nash equilibrium for all-pay auctions with general price externalities. The proof is based on Schauder's fixed-point theorem, which does not require a finite-dimensional set for the distribution functions. This result has potential applications in various economic contexts, including charity mechanisms and contests where relative performance is a critical factor.

Unfortunately, there is no closed form solution to this problem without providing a specific form of the externality functions. Employing a numerical approach, combined with laboratory experiments, could help uncover how non-linearity impacts the revenue performance of all-pay auctions. Investigating the uniqueness of equilibria is another essential avenue for future research. In the two-bidder case, the equilibrium can be characterized by a first-order condition that corresponds to a Fredholm equation. Kanwal (1971) provides a sufficient condition for uniqueness, $\sup\limits_{x\in[0,\bar{x}]} \int_0^{\bar{x}} \left| \dfrac{\partial h_i}{\partial x_i}(x_i, x_j) \right| dx_j < 1$. However, this condition is restrictive and lacks relevance for practical economic analysis.
To sum up, important directions for future research include addressing questions related to the uniqueness of equilibria and employing numerical methods to explore equilibrium properties. These efforts could yield significant insights for applications in fundraising mechanisms and competitive contests.

This paper also highlights an important practical issue concerning the use of all-pay auctions for charitable purposes. While theoretical results suggest that all-pay auctions are optimal mechanisms for fundraising (Goeree et al., 2005, Engers and McManus, 2007), empirical evidence from field experiments indicates otherwise (Carpenter et al., 2008, Onderstal et al., 2013). A possible explanation for this discrepancy lies in the assumption of linear bidder preferences regarding charitable giving. This contribution raises a novel question: what are the optimal mechanisms for fundraising when preferences exhibit non-linear externalities? Moreover, it prompts an analysis of the relative performance of all-pay auctions compared to other mechanisms commonly employed in practice, such as lotteries and voluntary contributions. This also opens up further investigation into the role of information (Szech, 2011*b*), including asymmetric distribution and opportunities for information sharing (Szech, 2011*a*).

5 Appendix

In this section, I provide a sketch of the proof showing the non-existence of a pure-strategy bidding equilibrium in all-pay auctions. For simplicity, I focus on the two-bidder case to make the argument easier to follow.

Assume $x_i \geqslant x_j$. Two cases arise:

1. Case 1: Bidder j can overbid. Her profitable deviation is $x_i + \varepsilon$ for $\varepsilon > 0$, such that
$$v_j - (x_i + \varepsilon) + h_j(x_i, x_i + \varepsilon) \geqslant -x_j + h_j(x_i, x_j).$$
In this case, $x_i \geqslant x_j$ cannot hold.
2. Case 2: Bidder j cannot overbid. Her best response is to bid zero, as
$$h_j(0, x_i) > -x_j + h_j(x_j, x_i),$$
given that $\frac{\partial h_j}{\partial x_j}(x_j, x_i) < 1$. In this scenario, bidder i's best response is to bid $\varepsilon > 0$. Hence, this leads to instability and the non-existence of a pure-strategy Nash equilibrium.

Additionally, in the general case described by assumption $A4$, where $\frac{\partial h_i}{\partial x_i}(x_i, \boldsymbol{x}_{-i}) \leqslant 1$ for all $i = 1, \ldots, n$, there is no pure-strategy Nash equilibrium in many situations. To clarify, I focus again on the two-bidder case. Suppose both bidders maximize $v_i - x_i + h_i(x_i, x_j)$ for $i = 1, 2, i \neq j$, choosing $(\tilde{x}_1, \tilde{x}_2)$ such that
$$\frac{\partial h_1}{\partial x_1}(\tilde{x}_1, \tilde{x}_2) = \frac{\partial h_2}{\partial x_2}(\tilde{x}_2, \tilde{x}_1) = 1.$$
If both bidders benefit from the same externality functions, i.e., $h_1 = h_2 \equiv h$, they will bid symmetrically: $\tilde{x}_1 = \tilde{x}_2 = \tilde{x}$, resulting in a payoff of
$$\frac{v_i}{2} - \tilde{x} + h(\tilde{x}, \tilde{x}) \quad \text{for } i = 1, 2.$$

Now, consider the case where $v_1 > v_2$ (such that bidders do not have the same maximum bid). Here, bidder i is always better off by overbidding $x_i = \tilde{x} + \varepsilon$ for $\varepsilon > 0$ and $\tilde{x}_1 = \tilde{x}_2 = \tilde{x}$ cannot be an equilibrium.[21] Using Taylor's theorem around the point $\tilde{x}$, we find
$$h(\tilde{x} + \varepsilon, \tilde{x}) = h(\tilde{x}, \tilde{x}) + \varepsilon \frac{\partial h}{\partial x_1}(\tilde{x}, \tilde{x}) + o(\varepsilon),$$

[21] If $v_1 = v_2 = v$, both bidders have the same maximum bid $\bar{x}$. The unique possible symmetric bidding equilibrium in pure strategies, $\tilde{x}_1 = \tilde{x}_2 = \tilde{x}$, is the highest $\tilde{x}$ such that $\tilde{x} \leqslant \bar{x}$ and $\dfrac{\partial h}{\partial x_i}(\bar{x}, \bar{x}) = 1$.

with $\lim_{\varepsilon \to 0} o(\varepsilon) = 0$. A bid of $\tilde{x} + \varepsilon$ leads to a payoff of

$$v_i - \tilde{x} - \varepsilon + h(\tilde{x}, \tilde{x}) + \varepsilon \frac{\partial h}{\partial x_1}(\tilde{x}, \tilde{x}) + o(\varepsilon).$$

Therefore, as $\frac{\partial h}{\partial x_1}(\tilde{x}, \tilde{x}) = 1$, overbidding by ε provides a higher payoff to bidder i. Bidder j's best response is either to overbid $\tilde{z}$, if

$$v_j - \tilde{z} + h(\tilde{z}, \tilde{x} + \varepsilon) > -\tilde{x} + h(\tilde{x}, \tilde{x} + \varepsilon),$$

or to underbid, choosing $\tilde{y}$. Excluding the case where $\frac{\partial h}{\partial x_j}(x_j, \tilde{x} + \varepsilon) = 1$ for all $x_j < \tilde{x}$, there exists some value $\tilde{y}$ such that

$$\frac{\partial h}{\partial x_j}(\tilde{y}, \tilde{x} + \varepsilon) < 1.$$

Thus, bidder j will underbid, choosing the smallest possible value, $\tilde{y}$.[22] Furthermore, excluding the case where $\frac{\partial h}{\partial x_i}(x_i, \tilde{y}) = 1$ for all $x_i \in (\tilde{y}, \tilde{x} + \varepsilon]$, there is a value $\tilde{y} + \kappa$ with $\kappa > 0$ such that bidder i will underbid. If v_2 is sufficiently high, bidder j will again deviate by proposing $\tilde{x}$. Hence, this is unstable and no pure-strategy Nash equilibrium exists. This result of the potential non-existence of bidding equilibria in pure strategies can be extended to heterogeneous externality functions with a similar reasoning.

References

Amegashie, J. (2001), 'An all-pay auction with a pure-strategy equilibrium', *Economics Letters* **70**, 79–82.

Amir, R. (1996), 'Continuous stochastic games of capital accumulation with convex transitions', *Games and Economic Behavior* **15**(2), 111–131.

Anderson, S. P., Goeree, J. K. and Holt, C. A. (1998), 'Rent seeking with bounded rationnality: An analysis of the all-pay auction', *Journal of Political Economy* **106**, 828–853.

Andreoni, J. (1989), 'Giving with impure altruism: Applications to charity and ricardian equivalence,', *Journal of Political Economy* **97**, 1447–1458.

Balbus, Ł., Jaśkiewicz, A. and Nowak, A. S. (2015), 'Stochastic bequest games', *Games and Economic Behavior* **90**, 247–256.

Baye, M., Kovenock, D. and de Vries, C. (1996), 'The all-pay auction with complete information', *Economic Theory* **8**(2), 291–305.

Becker, J. G. and Damianov, D. S. (2006), 'On the existence of symmetric mixed strategy equilibria', *Economics Letters* **90**, 84–87.

Bos, O. (2016), 'Charity auctions for the happy few', *Mathematical Social Sciences* **79**, 83–92.

Bos, O. and Ranger, M. (2018), 'All-pay auctions with endogenous rewards', *Annals of Economics and Statistics* **129**, 127–144.

[22] Indeed, $-\tilde{y} + h(\tilde{y}, \tilde{x} + \varepsilon) > -\tilde{x} + h(\tilde{x}, \tilde{x} + \varepsilon)$.

Carothers, N. L. (2000), *Real Analysis*, Cambridge University Press.

Carpenter, J., Homes, J. and Matthews, P. H. (2008), 'Charity auctions: A field experiment', *Economic Journal* **118**, 92–113.

Carpenter, J., Homes, J. and Matthews, P. H. (2010), 'Endogenous participation in charity auctions', *Journal of Public Economics* **94**, 921–935.

Chowdhury, S. (2017), 'The all-pay auction with non-monotonic payoff', *Southern Economic Journal* **84**(2), 375–390.

Curtat, L. O. (1996), 'Markov equilibria of stochastic games with complementarities', *Games and Economic Behavior* **17**(2), 177–199.

Damianov, D. and Peeters, R. (2018), 'Prize-based mechanisms for fund-raising: Theory and experiments', *Economic Inquiry* **56**(3), 1562–1584.

Engers, M. and McManus, B. (2007), 'Charity auction', *International Economic Review* **48**(3), 953–994.

Ettinger, D. (2010), 'Bidding among friends and enemies with symmetric information', *Journal of Institutional and Theoretical Economics* **166**(2), 365–385.

Fey, M. (2008), 'Rent-seeking contests with incomplete information', *Public Choice* **135**(3), 225–236.

Fudenberg, D. and Tirole, J. (1991), *Game Theory*, MIT Press, Cambridge, MA.

Goeree, J. K., Maasland, E., Onderstal, S. and Turner, J. L. (2005), 'How (not) to raise money', *Journal of Political Economy* **113**(4), 897–918.

Hillman, A. and Riley, J. (1989), 'Politically contestable rents and transfers', *Economics and Policy* **1**, 17–39.

Jehiel, P. and Moldovanu, B. (1996), 'Strategic nonparticipation', *RAND Journal of Economics* **27**(1), 84–98.

Kanwal, R. P. (1971), *Linear Integral Equations : Theory and Technique*, Academic Press, New York.

Klose, B. and Kovenock, D. (2015), 'The all-pay auction with complete information and identity-dependent externalities', *Economic Theory* **59**(1), 1–19.

Konrad, K. (2006), 'Silent interests and all-pay auctions', *International Journal of Industrial Organization,* **24**, 701–713.

Krakel, M., Szech, N. and von Bieberstein, F. (2014), 'Externalities in recruiting', *Journal of Economic Behavior & Organization* **107**, 123–135.

Monteiro, P. K. and Page Jr, F. H. (2007), 'Uniform payoff security and nash equilibrium in compact games', *Journal of Economic Theory* **134**(1), 566–575.

Onderstal, S., Schram, A. and Soetevent, A. R. (2013), 'Bidding to give in the field', *Journal of Public Economics* **105**(C), 72–85.

Prokopovych, P. and Yannelis, N. C. (2014), 'On the existence of mixed strategy nash equilibria', *Journal of Mathematical Economics* **52**(87-97).

Reny, P. J. (1999), 'On the existence of pure and mixed strategy nash equilibria in discontinuous games', *Econometrica* **67**(5), 1029–1056.

Reny, P. J. (2020), 'Nash equilibrium in discontinuous games', *Annual Review of Economics* **12**, 439–470.

Sacco, D. and Schumtzler, A. (2008), All-pay auctions with negative prize externalities: Theory and experimental evidence. University of Zurich, Socioeconomic Institute, Working Paper No. 0806.

Siegel, R. (2009), 'All-pay contests', *Econometrica* **77**(1), 71–92.

Siegel, R. (2014), 'Contests with productive effort', *International Journal of Game Theory* **43**(3), 515–523.

Simon, L. K. and Zame, W. R. (1990), 'Discontinuous games and endogenous sharing rules', *Econometrica* **58**(4), 861–872.

Stokey, N. L., Lucas, R. E. and Prescott, E. C. (1989), *Recursive Methods in Economic Dynamics*, Harvard University Press.

Szech, N. (2011*a*), Asymmetric all-pay auctions with two types. University of Bonn, Discussion paper.

Szech, N. (2011*b*), 'Optimal advertising of auctions', *Journal of Economic Theory* **146**(6), 2596–2607.