How Lotteries Outperform Auctions*

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Abstract

In their recent paper Goeree et al. (2005) determine that all-pay auctions are better for fundraising activities than lotteries. We show that the introduction of asymmetry among participants with complete information could reverse this result. Complete information seems well suited to some charity environments.

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1 Introduction

In their recent paper Goeree et al. (2005) analyze an independent private values model with financial externalities independent of the winner's identity and show that the first-price all-pay auction (hereafter all-pay auction) outperforms lotteries and winner-pay auctions.[1] All-pay auctions combine two effects. On the one hand, like winner-pay auctions, all-pay auctions are efficient. Yet, like lotteries, they are associated with positive externalities – or a return – from the losers' bids. However, lotteries seem to be used more frequently by fundraisers than auctions. Maybe some element, as asymmetry or heterogeneity among participants, missing in Goeree et al.'s analysis could explain this common use of lotteries.

In this note we focus on all-pay auctions and lotteries as their revenue is not bounded – in contrast to winner-pay auctions.[2] Is it still true that all-pay auctions are better at raising

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[1] They prove that all-pay auctions outperforms lotteries with independent private values model and symmetric bidders in Proposition 2 and in a symmetric complete information environment in footnote 4, page 8. They also show that the lowest-price all-pay auction with the suitable fees and reserve price is an optimal fundraising mechanism. Engers and McManus (2007) determine other optimal auctions for charity.

[2] See Bos (2008) for a comparison between all-pay auctions and winner-pay auctions with asymmetric bidders under complete information.

money for charity with asymmetric participants? Agents can be asymmetric in different ways. By instance, their types can be drawn from different distributions with incomplete information or their types can be strictly ordered with complete information. Both complete information and incomplete information are well suited to different kinds of fundraising environments. For example, complete information could occur in some charity dinners among people of the same social class who know each other well and also in some voluntary social organizations. On the other hand, incomplete information is found in some fundraising activities on the Internet.[3] As few results are available on asymmetric all-pay auctions in an independent private values model, we investigate complete information. We show that the introduction of asymmetry among either valuations, altruistic parameters or both with complete information can reverse this result of Goeree et al. (2005).

This paper is organized as follows. In the next section we give a presentation of the framework and the raffle. In section 3 we compare lottery and all-pay auction revenues. We do this using Bos' (2008) results on all-pay auctions for charity with complete information.

2 The Framework

We consider that a fundraiser has one item she is going to sell in a local community of n people by means of a lottery and an all-pay auction. In this community, people know each other well so the valuation of participant i, denoted v_i, is common knowledge. As the item is sold to collect money for a charity purpose, participants get an additional benefit from the revenue raised. Let α_i denote the altruism parameter or the return to participant i of the fundraising activity which is also common knowledge. As in Goeree et al. (2005), participants are not completely altruistic, which means that their altruism parameter is strictly less than one - and positive.[4]

To the best of our knowledge, Morgan (2000) was the first to study lotteries as a fundraising mechanism. Yet, unlike Morgan (2000) here the asymmetry is not only on the altruism parameters but also on the valuations. We denote x_i the number of tickets bought by player i such that the revenue collected is $R^{LOT} \equiv \sum_{i=1}^{n^p} x_i$ with $n^p \leq n$ being the number of active participants. Thus, the expected utility of i is

$$\mathbb{E}U_i(x_i, x_{-i}) = v_i P_i(x_i, x_{-i}) - x_i + \alpha_i \sum_{j=1}^{n} x_j \tag{1}$$

where $P_i(x_i, x_{-i}) = \frac{x_i}{\sum_{i=1}^{n} x_i}$ is the probability of winning for potential participant i and $\alpha_i \sum_{i=j}^{n} x_i$ the return he gets from the amount raised. Note that the all-pay auction leads to the same kind of expected utility given that x_i is a bid and $P_i(x_i, x_{-i})$ is a cumulative

[3] See for example, the eBay charity auction website *Giving Works*, ebaygivingworks.com, and the charity lotteries held on charityfacts.org.

[4] This assumption is straightforward to the framework of Bos (2008) that we used to compare the revenue with all-pay auctions. Technically, this assumption means that the utility of participants decreases if the amount of money they spend (their bid or the number of tickets bought) increases.

distribution function. As $\sum_{j=1,j\neq i}^{n} x_j$ is independent of the number of tickets bought by i and α_i is inferior to 1, the expected payoff (1) is strategically equivalent to

$$t_i P_i(x_i, x_{-i}) - x_i \text{ with } t_i = \frac{v_i}{1-\alpha_i}$$

which is the usual expected payoff without any charity component for the lottery and the all-pay auction (where t_i is the participant i's type). As valuations and altruistic parameters are common knowledge the types of the participants and the ranking of the types $t_1 > t_2 > t_3 \geq ... \geq t_n$ are also common knowledge.[5] The equilibrium in the all-pay auction is already pointed out by Bos (2008), then we will focus only on the lottery in the rest of this section. As our lottery is similar to a Tullock contest (Tullock, 1980), the equilibrium is unique and there are at least two participants in the lottery: these are the two members of the community with the two highest types who take part. Moreover, if a participant with a type t_i is active, then all the participants with higher types are also active. The proof for these results is omitted as already done by Corchón (2007) for the number of tickets bought and the revenue raised and by Meland and Straume (2007) for the number of participants.

Proposition 1. *The lottery has a unique Nash equilibrium such that the number of tickets bought by the participants is given by* $x_i = \begin{cases} \frac{n^p-1}{\sum_{j=1}^{n^p} 1/t_j}\left(1 - \frac{n^p-1}{t_i\sum_{j=1}^{n^p} 1/t_j}\right) & \forall i \leq n^p \\ 0 \textit{ otherwise} \end{cases}$ *and the revenue raised is* $R^{LOT} = \frac{n^p-1}{\sum_{i=1}^{n^p} 1/t_i}$ *with* n^p *the highest integer of* $m \in \{2, ..., n\}$ *which satisfies* $m \leq 2 + t_m \sum_{i=1}^{m-1} \frac{1}{t_i}$.

3 Revenue Comparison

In the following, we use Bos' (2008) results on all-pay auctions for charity to compare them with our result on lotteries. Bos determines the unique Nash equilibrium in the all-pay auctions with financial externalities. It is a mixed strategies equilibrium where only the two bidders with the highest types participate. Then, the all-pay auction expected revenue is not bounded, as is the revenue of the lottery. The results are summed up in the following table where $\mathbb{E}R^{AP}$ is the expected revenue of the all-pay auction:

[5] This ranking is relevant with the purpose of this paper which is to determine the consequence of heterogeneity. Moreover, consider at least the three highest types are strictly ordered avoid the multiplicity of equilibria in the all-pay auction.

Ranking of types	R^{LOT}	$\mathbb{E}R^{AP}$
$t_1 = ... = t_n \equiv t$	$\frac{n-1}{n}t$	t
$t_1 > t_2 > ... \geq t_n$	$(n^p - 1)\frac{1}{\sum_{i=1}^{n^p} 1/t_i}$	$\frac{t_2}{2}\left(\frac{t_2}{t_1}+1\right)$

Table 1: Revenue and expected revenue for each design

Suppose that participants have the same types. Then, it follows that all-pay auctions lead to higher revenues than lotteries for charity. In fact, we find the same qualitative results as Goeree et al. (2005) and confirm the ones of Orzen (2005) who compared all-pay auctions and lotteries with complete information in a different framework. However, as seen in the next proposition, this result does not hold when the asymmetry between the active participants is strong enough.[6]

Proposition 2. *Lottery is better at raising money for charity than all-pay auction if and only if the participants with the two highest types are asymmetric enough.*

Proof. For $n^p = 2$, $R^{LOT} > \mathbb{E}R^{AP}$ is true if and only if $\frac{1}{\frac{1}{t_1}+\frac{1}{t_2}} > \frac{1}{2}\frac{t_2}{t_1}(t_2+t_1)$

$$\Leftrightarrow 1 > \frac{t_2^2}{t_1^2} + 2\frac{t_2}{t_1} \tag{2}$$

$$\Leftrightarrow t_1 - t_2 > \frac{t_2}{t_1}(t_2+t_1) \tag{3}$$

The inequality (2) is depicted in the positive area of Figure 1. In this figure $1 - \frac{t_2^2}{t_1^2} - 2\frac{t_2}{t_1}$ is drawn for all (t_1, t_2) such that $\frac{t_2}{t_1}$ varies from 0 to 1 in the horizontal axis. Cases $\frac{t_2}{t_1} \to 0$ and $\frac{t_2}{t_1} \to 1$ are extreme and represent the highest level of asymmetry and full symmetry. Thus, Figure 1 guaranties the existence of vector values (t_1, t_2) such that inequality (3) is satisfied. Remark the higher is the asymmetry the better is the revenue performance of the lottery relatively to the all-pay auction.

For $n^p = 3$, $R^{LOT} > \mathbb{E}R^{AP}$ is true if and only if $2\frac{1}{\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}} > \frac{1}{2}\frac{t_2}{t_1}(t_2+t_1)$

$$\Leftrightarrow 3 > \frac{t_2^2}{t_1^2} + 2\frac{t_2}{t_1} + \frac{t_2}{t_1}\frac{t_1+t_2}{t_3}$$

$$\Leftrightarrow 3t_1^2 - t_2^2 - 2t_1t_2 > t_1t_2\frac{t_1+t_2}{t_3}$$

$$\Leftrightarrow t_1 - t_2 > \frac{t_1t_2}{t_3(3t_1+t_2)}(t_2+t_1) \tag{4}$$

[6] A related result has been shown by Fang (2002). Unlike in our setup, he compares an all-pay auction and a lottery without a charity component. He investigates this framework with all weakly ordered types which leads to a multiplicity of equilibria in the all-pay auction. Then he determines (only) a sufficient condition for the revenue comparison for more than two agents.

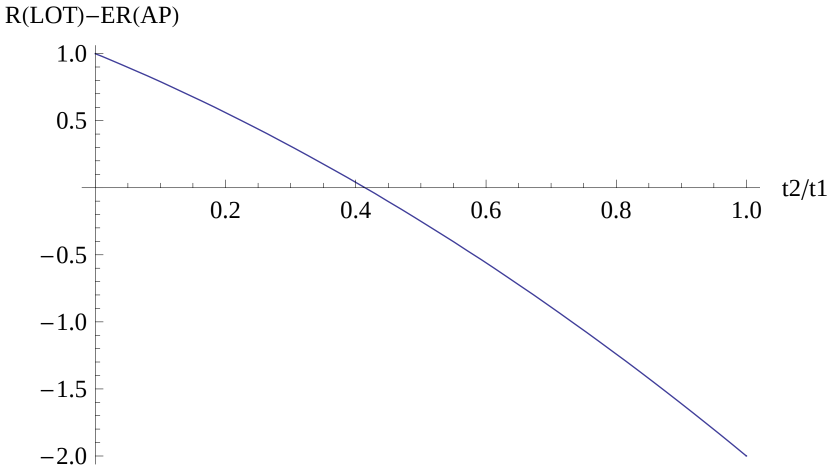

Figure 1: Comparison R^{LOT} and $\mathbb{E}R^{AP}$ for different levels of asymmetry.

Note we can find numerical examples such that either (4) is satisfied or not. Indeed, $(t_1, t_2, t_3) = (25, 5, 4)$ guarantees inequality (4) holds, while $(t_1, t_2, t_3) = (10, 5, 4)$ does not.

Finally, for $n^p > 3$, $R^{LOT} > \mathbb{E}R^{AP}$ is true if and only if $\dfrac{n^p - 1}{\sum_{i=1}^{n^p} \frac{1}{t_i}} > \dfrac{1}{2}\dfrac{t_2}{t_1}(t_2 + t_1)$

$$\Leftrightarrow 2(n^p - 1)t_1 > \frac{t_2^2}{t_1} + 2t_2 + t_1 + t_2(t_2 + t_1)\sum_{i>2}^{n^p} \frac{1}{t_i}$$

$$\Leftrightarrow 2(t_1 - t_2) > 2(-n^p + 3)\frac{t_2^2}{t_1} - (2n^p - 5)\frac{t_1^2 - t_2^2}{t_1}(t_2 + t_1)\sum_{i>2}^{n^p} \frac{1}{t_i}$$

$$\Leftrightarrow (t_1 - t_2)\frac{(2n^p - 3)t_1 + (2n^p - 5)t_2}{t_1} > 2(-n^p + 3)\frac{t_2^2}{t_1} + t_2(t_2 + t_1)\sum_{i>2}^{n^p} \frac{1}{t_i}$$

$$\Leftrightarrow t_1 - t_2 > \frac{1}{\Pi_{i>2}^{n^p} t_i}\frac{t_2}{(2n^p - 3)t_1 + (2n^p - 5)t_2}\left(-2(n^p - 3)t_2\Pi_{i>2}^{n^p} t_i + t_1(t_2 + t_1)\sum_{k>2} \Pi_{\substack{i>2\\ i\neq k}}^{n^p} t_i\right) \tag{5}$$

As above, we can find numerical examples such that either (5) is satisfied or not. For examples, for $n^p = 5$, $(t_1, t_2, t_3, t_4, t_5) = (25, 5, 4, 3, 2)$ guarantees inequality (5) holds rather than $(t_1, t_2, t_3, t_4, t_5) = (10, 5, 4, 3, 2)$ does not. ■

If our framework is well suited to certain charity settings (*e.g.* dinners held by a local Rotary Club), the introduction of asymmetry among participants contradicts Goeree et al.'s (2005) qualitative results. Note that alternatively of the proof of Proposition 2, it could be shown the lottery raises more money than the all-pay auction with two bidders if $t_1 > (1+\sqrt{2})t_2$.[7] This restriction to two bidders leads to the same necessary and sufficient condition

[7] Indeed if $t_1 > (1 + \sqrt{2})t_2$, $t_1^2 - 2t_1t_2 - t_2^2 > 0$ and then $R^{LOT} > \mathbb{E}R^{AP}$. I am grateful to an anonymous referee for pointing out this result.

as in Fang (2002).

4 Conclusion

In this paper we show that lotteries could be better at raising money for charity than all-pay auctions when participants are asymmetric enough. Moreover, as lottery revenues are not bounded, this mechanism seems more appropriate than auctions for fundraising activities.

This work could be rounded out by a laboratory experiment. Only two lab experiments have been led to compare lotteries and all-pay auctions. Onderstal and Schram (2009) compared lotteries, first-price all-pay and winner-pay auctions within the framework of Goeree et al. (2005) while Orzen (2005) runs an experiment with a complete information framework for symmetric participants. The former confirms the theory while the latter is inconclusive. As asymmetry can change the theoretical results it would be interesting to conduct new experiments with asymmetric participants.

Finally, this paper leaves open for future research the question of fundraising mechanisms with asymmetric participants under incomplete information.

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