Received: 30 September 2017 | Accepted: 12 October 2019

DOI: 10.1111/jpet.12406

ORIGINAL ARTICLE

JOURNAL OF PUBLIC ECONOMIC THEORY WILEY

Charitable asymmetric bidders

Olivier Bos[1,2,3]

[1]LEMMA, Panthéon-Assas University, Paris, France

[2]Institut Universitaire de France (IUF), Paris, France

[3]ZEW - Leibniz Centre for European Economic Research, Mannheim, Germany

Correspondence
Olivier Bos, LEMMA, Panthéon-Assas University, 4 rue Blaise Desgoffe, 75006 Paris, France.
Email: olivier.bos@u-paris2.fr

Funding information
European Science Foundation, Grant/Award Number: PGPPE - 2373; FP6 People: Marie-Curie Actions, EST-Early-Stage Training

Abstract

Recent papers show that the all-pay auction is better at raising money for charity than the first-price auction with symmetric bidders under incomplete information. Yet, this result is lost with sufficiently asymmetric bidders under complete information. In this paper, we consider a framework on charity auctions with asymmetric bidders under some incomplete information. We find that the all-pay auction still raises more money than the first-price auction. Thus, the all-pay auction should be seriously considered when one wants to organize a charity auction.

1 | INTRODUCTION

Fundraising activities for charitable purposes have become increasingly popular. One reason is the growing number of nongovernmental organizations with humanitarian or social purposes. Another one is the decrease of government participation in culture, education, and related activities. The purpose of these associations is either the development and promotion of culture or aid and humanitarian services. Even in France, a country without any fundraising tradition, some organizations began to appear, such as the *French Association of Fundraisers*[1] in 2007.

Commonly used mechanisms to raise money are voluntary contributions, lotteries, and auctions. Even though most of the fundraisers still use voluntary contributions,[2] auctions are increasingly used. Indeed, for some special events or particular situations, auctions provide a

[1]http://www.fundraisers.fr/

[2]There is further evidence of this phenomenon on the Internet with the emergence of sites, such as http://www.JustGive.org.

particular atmosphere. The popularity of auctions for charity purposes can also be observed by the increase in Internet websites offering the sale of objects and donating a part of their proceeds to charity. Well-known examples include *Yahoo!* and *Giving Works* of *eBay*. Many others have been created, such as the *Pass It On Celebrity Charity Auction*[3] in 2003, where celebrities donated objects whose sale revenue contributed to a "charity of the month". We can also cite *cMarket Charitable Auctions Online*[4] created in 2002 and selected as a charity vehicle by more than 930 organizations.

Therefore, given the well-developed and wide theoretical literature on altruism and charitable fundraising (e.g., Andreoni, 1989, 1998, 2004), there is a growing and recent research on charity auctions.[5] Goeree, Maasland, Onderstal, and Turner (2005) and Engers and McManus (2007) investigate an independent private value's model and show that all-pay auctions are better at raising money for charity than winner-pay auctions. Moreover, Schram and Onderstal (2009) lead a lab experiment and confirm these theoretical results. However, Carpenter, Homes, and Matthews (2008) run a field experiment in four American preschools. In their experiments the ranking of the revenues is reversed. They attribute this result to the unfamiliarity of the participants to the mechanism and endogenous participation (see Carpenter, Homes, & Matthews, 2010 for a theoretical investigation of the endogenous participation). In addition, we can also investigate this question in a situation where people are different in the sense that they do not have the same beliefs. Indeed, Goeree et al. (2005) and Engers and McManus (2007) assume that bidders have the same altruism parameter and valuations are drawn from the same distribution. Bos (2016) provides an answer with complete information. He investigates a model with complete information and heterogeneity on the bidders' values, and shows that when the asymmetry among bidders is strong enough, the ranking of the expected revenues is affected. In particular, winner-pay auctions outperform all-pay auctions. Damianov and Peeters (2018) confirm in a lab experiment the nonoptimality of all-pay auctions in a complete information framework.

The point of this paper is then to determine, whether all-pay auctions are still better at raising money for charity when bidders are asymmetric under some incomplete information (provided by a uniform distribution). To do so, we compare the first-price auction and the (first-price) all-pay auction. The former is the most used sealed-bid auction in practice and the latter raises theoretically more money for charity than winner-pay auctions in a symmetric incomplete information framework. We do not consider the optimal auction determined by Goeree et al. (2005), the lowest-price all-pay auction, because of its difficult implementation and potential participants misunderstanding.[6] If we conclude that all-pay auctions are still better with asymmetric bidders and our specific incomplete information setting with uniformly distributed values, we should seriously consider implementing all-pay auctions to raise money for charity in some environments. Indeed, to the best of our knowledge, all-pay auctions have never been implemented in real life for charity purposes. However, it seems easy to do. For example, every bidder could buy a number of tickets simultaneously as in a raffle. Contrary to a raffle, though, the winner will be the buyer with the highest number of tickets in hand.

[3]http://www.passitonline.org/

[4]http://www.cmarket.com/

[5]For a good understanding on prosocial behavior see Munoz-Herrera and Nikiforakis (2019), who provide a thorough overview on the theoretical, lab, and field experimental works of James Andreoni.

[6]Difficulties occurred on the field to implement the first-price all-pay auction (Carpenter et al., 2008; Onderstal, Schram, & Soetevent, 2013) and should be worst with the lowest-price all-pay auction.

In charity auctions, bids depend upon two parameters: their valuation for the item sold and their altruism or sensitivity to the charity purpose. In this paper we consider valuations drawn with the same distribution in an independent private value's model. Therefore bidders are asymmetric in their altruism parameter which are common knowledge. As in Bulow, Huang, and Klemperer (1999) and Wasser (2013) this framework provides some advantages, mainly to avoid the limited results of asymmetric auctions with incomplete information. In the usual asymmetric auction literature, valuations are drawn from different distributions. Changing these distributions can modify the ranking of the expected revenues among different auction designs (e.g., see Krishna, 2009). Maskin and Riley (2000), de Frutos (2000), and Cantillon (2008) determine the ranking of the expected revenues between the first-price and the second-price auctions given some conditions on the probability distributions.

Bulow et al. (1999) paper is probably the closest to the present analysis. They investigate first-price and ascending auctions with a two-bidder common value setting. Each bidder receives an independent uniformly distributed signal, which contributes to a common value, and a parameter which can be interpreted as an altruism parameter for charity purpose. These latter parameters are asymmetric and common knowledge. Although they apply this framework to toeholds and takeovers, it is well suited for charity. In their analysis, they determine that when these parameters are asymmetric and small enough, the revenue ranking could be reverse (relatively to the symmetric case). Therefore the first-price auction outperforms in terms of expected revenue the ascending auction.[7] Unlike them, we compare first-price to all-pay auctions in an independent private value's model. The only other papers on asymmetric auctions with these kinds of externalities are de Frutos (2000) and Wasser (2013). de Frutos (2000) compares the first-price and the second-price auctions with altruism parameters equal to $1/2$ and different bidders' value distributions. Her framework is quite different to ours as she does not investigate all-pay auctions and the asymmetry concerns bidders values instead of altruism parameters. However, dividing our all-pay auction by 1 minus the bidder's altruism parameter leads to the all-pay auction in her framework with uniform distributions.[8] Wasser (2013) investigates $k+1$-price winner-pay auctions with asymmetry on the altruism parameters. Yet, he does not compare the expected revenue among the auction design but focuses on the performance of auctions as mechanisms for partnership dissolution. Thus our both papers are related through the existence and uniqueness of the first-price auction but differs on problems raised and results determined. Furthermore, Lu (2012) determines an optimal mechanism with asymmetric altruism parameters in which all losers pay a fraction of the winner payment and then is a kind of all-pay auction. The set of mechanisms investigated is contingent on the positive bidders' reservation utilities and therefore requires a nonparticipation threat, such as the seller can cancel the auction. However in practice, a such threat is not observed, the auction is not dissolved if some bidders do not participate. In our positive approach, we compare the most popular sealed-bid auction used for charity, the first-price auction, and the easiest all-pay auction to implement, without any nonparticipation threat. It is then an interesting complement of the Lu's normative approach, and determines if all-pay mechanisms, in their simplest form, should replace the first-price auction.

The paper is organized as follows. Section 2 introduces the formal setting, a two-bidder independent private value's model with asymmetric altruism parameters. Section 3 and 4, respectively, characterized the bidding equilibrium strategies for the all-pay auction and the

[7]Moreover one of their main contributions is also to show how first-price and ascending auctions are affected differently by the winners curse.

[8]This is not true for the first-price auction.

first-price auction. Section 5 provides the revenue comparison of these auctions and determines that all-pay auction still outperforms first-price auction independently of level of asymmetry in their altruism parameters. Section 6 concludes. Proofs not providing after the results are collected in the appendix.

2 | FORMAL SETTING

Suppose two bidders take part in an auction through a fundraising event, such as a charity dinner. Each bidder is risk neutral and cares about how much she pays as well as her competitor pays in the auction. Indeed, as the amount of money will be used for a charity purpose, the bidders include in their utility function the bids paid. Thus, their bidding functions depend of two parameters: Their valuation of the object sold and their altruism or their interest for the charity purpose that the auction should finance. The more a bidder is sensitive to the charity event the higher this parameter will be. Denote as v_i the valuation and as a_i bidder i's altruism parameter. Bidder valuations v_1, v_2 are independently and identically distributed and we assume them to uniform distributions on $[0, \bar{v}]$ with $\bar{v} \geq 1$. Moreover, the altruism parameters are common knowledge and heterogeneous such that $a_1 > a_2 \geq 0$. Then, Bidder 1 has a higher preference for the charity purpose than Bidder 2. When a bidder takes part in a charity auction, she obtains a positive externality from the amount of money raised. Indeed, she hopes that the highest amount will be collected to finance the charity purpose. This is equivalent to a situation in which she would benefit from a percentage of the revenue collected as a return from the bids paid. In this paper we consider two auction designs: the all-pay auction, also called first-price all-pay auction, and the usual first-price auction.

In the all-pay auction the winner as well as the loser pays her own bid. Yet, each bidder receives an externality from her own bid as well as from her competitor's bid. Denote as $U_i^{\mathrm{A}}(v_i, b_i, b_j; a_i)$ the utility of bidder i

$$U_i^{\mathrm{A}}(v_i, b_i, b_j; a_i) = \begin{cases} v_i - b_i + a_i(b_i + b_j) & \text{if } b_i > b_j, \\ -b_i + a_i(b_i + b_j) & \text{if } b_i < b_j, \\ \dfrac{v_i}{2} - b_i + a_i(b_i + b_j) & \text{if } b_i = b_j. \end{cases} \quad (1)$$

In contrast, in the first-price auction the bidder with the highest bid is the winner and pays her own bid, whereas the loser does not pay anything. Contrary to the all-pay auction, here each bidder benefits from an externality only from the winner's bid which could be her own bid. Denote as $U_i^{\mathrm{F}}(v_i, b_i, b_j; a_i)$; the utility of bidder i follows

$$U_i^{\mathrm{F}}(v_i, b_i, b_j; a_i) = \begin{cases} v_i - b_i + a_i b_i & \text{if } b_i > b_j, \\ a_i b_j & \text{if } b_i < b_j, \\ \dfrac{v_i}{2} - \dfrac{b_i}{2} + a_i b_i & \text{if } b_i = b_j. \end{cases} \quad (2)$$

It is clear that the payment rule affects the returns that bidders obtain. In the all-pay auction, bidder i's utility is a function of her opponent's bid for each outcome of the auction. In the first-price auction, on the other hand, if the bidder i is the winner her payoff is independent of her opponent's bid.

Assumption (The limit of the bidders' altruism). $a_1 < 1$ in the all-pay auction and the first-price auction.[9]

The assumption states that bidders strictly prefer to keep their money for personal use rather than to spend it for the charitable purpose even if they win.

Bidder i's strategy is a function $\alpha(\cdot;a_i) : [0, \bar{v}] \to \mathbb{R}_+$ in the all-pay auction and a function $\beta(\cdot;a_i) : [0, \bar{v}] \to \mathbb{R}_+$ in the first-price auction which determines her bid for any value given her altruism parameter. Given a sensitivity level a_i different for each bidder, we focus on the asymmetric equilibria such that $\alpha(\cdot;a_i) \equiv \alpha_i(\cdot)$ and $\beta(\cdot;a_i) \equiv \beta_i(\cdot)$. However, as the bidders are distinguished only thanks to their altruism parameter, their equilibrium bidding functions would be symmetric in these parameters. Denote as $\varphi_i(\cdot) = \alpha_i^{-1}(\cdot)$ and $\phi_i(\cdot) = \beta_i^{-1}(\cdot)$ the inverse functions of bidder i's strategy functions given her altruism a_i.[10] Notice that (α_i, α_j) is a Bayesian Nash equilibrium such that it fulfills the first- and second-order conditions if and only if (φ_i, φ_j) also fulfills the first- and second-order conditions. The same relationship also holds in the first-price auction with (β_i, β_j) and (ϕ_i, ϕ_j).

3 | ALL-PAY AUCTION

Using (1) we can compute the expected payoff of bidder i

$$\mathbb{E}U_i^{\mathrm{A}}(v_i, b_i, \alpha_j; a_i) = \frac{v_i}{\bar{v}}\alpha_j^{-1}(b_i) - b_i + a_i(b_i + \mathbb{E}\alpha_j(v_j)). \tag{3}$$

To determine the effect of the altruism on the expected payoff we can divide (3) in two terms, the usual expected utility and the return from the charity purpose, κ_i^{A}. Then,

$$\mathbb{E}U_i^{\mathrm{A}}(v_i, b_i, \alpha_j; a_i) = \frac{v_i}{\bar{v}}\alpha_j^{-1}(b_i) - b_i + \kappa_i^{\mathrm{A}}(b_i, \alpha_j; a_i)$$

with $\kappa_i^{\mathrm{A}}(b_i, \alpha_j; a_i) = a_i(b_i + \mathbb{E}\alpha_j(v_j))$. Thus, if bidder i does not take account of the term κ_i^{A}, she would face the usual all-pay auction expected payoff.

Lemma 1. *The bidders' equilibrium strategies must be pure strategies that are continuous and strictly increasing functions in v_i.*

Lemma 2. *Minimum and maximum bids must be the same for both bidders so that $\alpha_1(0) = \alpha_2(0) = 0$ and $\alpha_1(\bar{v}) = \alpha_2(\bar{v}) = \bar{b}$.*[11]

In an all-pay auction, bidders care about their bids if they win as well as when they lose. In both cases, they get a positive return from their opponent's bid. Thus, their equilibrium bid depends on their own altruism parameter as well as on their competitor's. An immediate

[9]It follows that $a_2 < a_1 < 1$.

[10]It is established in Lemmas 1 and 3 that $\alpha_i(\cdot)$ and $\beta_i(\cdot)$ are strictly increasing functions.

[11]Notice that contrary to Amann and Leininger (1996) and Lu and Parreiras (2017) who investigated two-bidder all-pay auctions with asymmetric beliefs, in our model there is no mass bid at zero. Bidders have symmetric beliefs and are asymmetric in their altruism parameters, which are common knowledge and lower than 1. Therefore both bidders have the same beliefs about their rival to be a zero value and a zero value bidder gets the same losing and winning payoffs. The effect of asymmetry for a zero value is vanished and only the symmetric beliefs play a role.

consequence of Lemma 1 is that the inverse function of α_i, φ_i is increasing and differentiable almost everywhere on $[0, \bar{b}]$. Furthermore, $\varphi_i(0) = 0$ and $\varphi_i(\bar{b}) = \bar{v}$, where $\bar{b} = \alpha_1(\bar{v}) = \alpha_2(\bar{v})$.

To derive the equilibrium, we state here only the necessary condition. A sufficient condition is given in the appendix (Proof of Proposition 1). Differentiating (3) with respect to b_i it follows that

$$\varphi_1(b) = \bar{v}\frac{1 - a_1}{\varphi_2'(b)} \quad \text{for all } b \in (0, \bar{b}], \tag{4}$$

$$\varphi_2(b) = \bar{v}\frac{1 - a_2}{\varphi_1'(b)} \quad \text{for all } b \in (0, \bar{b}]. \tag{5}$$

Then, from (4) and (5) and using the boundary conditions $\varphi_i(0) = 0$ we get

$$\varphi_i(b)\varphi_j(b) = (1 - a_i)\bar{v}b + (1 - a_j)\bar{v}b \quad \text{for all } b \in (0, \bar{b}]. \tag{6}$$

As $\varphi_i(\bar{b}) = \bar{v}$ for all i, $\bar{b} = \frac{\bar{v}}{2 - a_1 - a_2}$ follows from (6). Then, for some level of the altruism parameters, bidders could submit a maximum bid higher than their valuation. Indeed, this would be the case if the sum of the altruism parameters is higher than 1. Moreover, if each altruism parameter is close to 1, the maximum bid would be infinite as in the case of symmetric bidders (see Goeree et al., 2005). Thus, revenue is not bounded and could potentially be infinite.

Using (5), for $i = 1, 2$ Equation (6) leads to

$$\varphi_i(b) = \frac{2 - a_i - a_j}{1 - a_j}\varphi_i'(b)b \quad \text{for all } b \in (0, \bar{b}]. \tag{7}$$

From this we obtain an explicit solution to the inverse bid functions which characterize the unique Bayesian Nash equilibrium[12] $(\varphi_1(\cdot), \varphi_2(\cdot))$:

$$\varphi_i(b) = \bar{v}^{\frac{1-a_i}{2-a_i-a_j}}((2 - a_i - a_j)b)^{\frac{1-a_j}{2-a_i-a_j}} \quad \text{for all } b \in (0, \bar{b}], \quad \text{for } i = 1, 2. \tag{8}$$

Proposition 1. *There exists a unique Bayesian Nash equilibrium* (α_1, α_2) *such that*

$$\alpha_i(v) = \frac{1}{\bar{v}^{\frac{1-a_i}{1-a_j}}}\frac{1}{2 - a_i - a_j}v^{\frac{2-a_i-a_j}{1-a_j}} \quad \text{for all } v \in [0, \bar{v}], \quad i = 1, 2 \quad \text{and} \quad i \neq j.$$

Obviously, for $a_1 = a_2 \equiv a$ we get the symmetric Nash equilibrium

$$\alpha_1(v) = \alpha_2(v) = \frac{1}{\bar{v}}\frac{1}{2(1 - a)}v^2.$$

[12]Let us assume there are two solutions to Equation (7), $\bar{\varphi}$ and $\underline{\varphi}$. It follows that $\frac{1 - a_j}{2 - a_i - a_j}\frac{1}{b} = \frac{\bar{\varphi}'(b)}{\bar{\varphi}(b)} = \frac{\underline{\varphi}'(b)}{\underline{\varphi}(b)}$ for all $b \in (0, \bar{b}]$. Then $\ln \mid \bar{\varphi}(b) \mid = \ln \mid \underline{\varphi}(b) \mid + c$ with $c \in \mathbb{R}$. $\bar{\varphi}(\bar{b}) = \underline{\varphi}(\bar{b})$ implies that $c = 0$ and therefore $\bar{\varphi}(b) = \underline{\varphi}(b)$ for all $b \in (0, \bar{b}]$. Hence a contradiction.

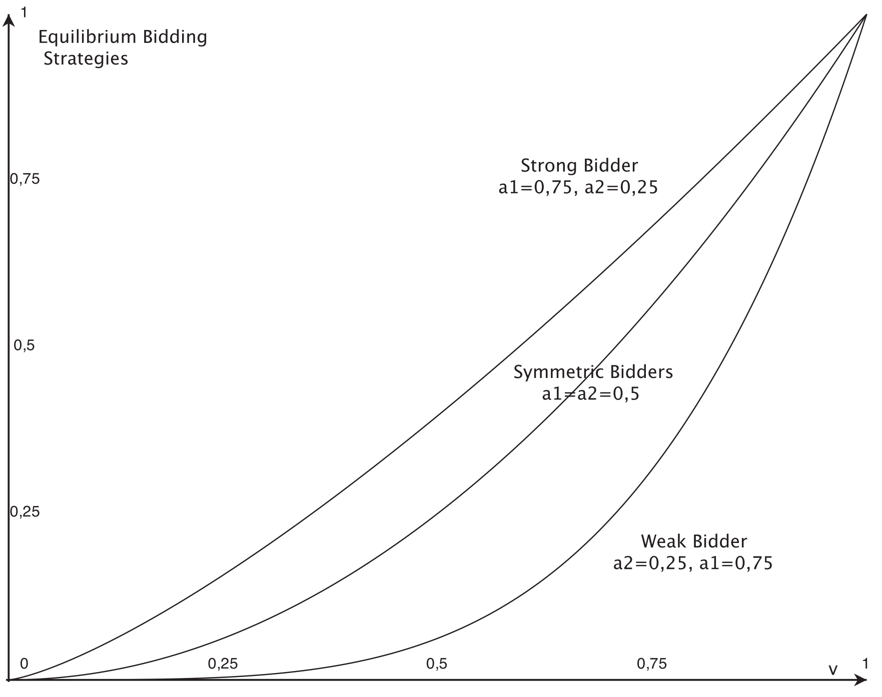

FIGURE 1 Equilibrium bidding strategies

The bidder i's equilibrium bidding strategy is increasing in her altruism parameter a_i. Indeed, the more she is sensitive to the charity purpose the more aggressively she will bid. However, the bidder i's equilibrium bidding strategy is not monotonic in her opponent's altruism parameter a_j. These results can be verified by computing the following derivatives:

$$\frac{\partial \alpha_i}{\partial a_i}(v; a_i, a_j) = \frac{v}{(2 - a_i - a_j)^2}\left(\frac{v}{\bar{v}}\right)^{\frac{1-a_i}{1-a_j}}\left(1 - \frac{2 - a_i - a_j}{1 - a_j}\ln\frac{v}{\bar{v}}\right) \geq 0,$$

$$\frac{\partial \alpha_i}{\partial a_j}(v; a_i, a_j) = \frac{v}{(2 - a_i - a_j)^2}\left(\frac{v}{\bar{v}}\right)^{\frac{1-a_i}{1-a_j}}\left(1 + (2 - a_i - a_j)\frac{1 - a_i}{(1 - a_j)^2}\ln\frac{v}{\bar{v}}\right).$$

However, it follows from Proposition 1 that a higher sensitivity to altruism leads to a higher aggressiveness. Figure 1 depicts the equilibrium bidding strategies for $a_1 = 0.75$, $a_2 = 0.25$, and $\bar{v} = 1$.

Corollary 1. *In the all-pay auction, the more altruistic bidder is the more aggressive one. More precisely, if $a_1 > a_2$, then $\alpha_1(v) > \alpha_2(v)$ for all $v \in (0, \bar{v})$.*

4 | FIRST-PRICE AUCTION

Using (2) we can then compute the expected payoff of bidder i

$$\mathbb{E}U_i^{\mathrm{F}}(v_i, b_i, \beta_j; a_i) = \frac{v_i - (1 - a_i)b_i}{\bar{v}}\beta_j^{-1}(b_i) + \frac{a_i}{\bar{v}}\int_{\beta_j^{-1}(b_i)}^{\bar{v}} \beta_j(v)\, dv. \quad (9)$$

Again, we can split the expected payoff in two terms. The first one is the expected payoff of the usual first-price auction and the second the return from the charity purpose, κ_i^{F}:

$$\frac{v_i - b_i}{\bar{v}}\beta_j^{-1}(b_i) + \kappa_i^{\mathrm{F}}(b_i, \beta_j; a_i)$$

with $\kappa_i^{\mathrm{F}}(b_i, \beta_j; a_i) = \frac{a_i}{\bar{v}}(b_i\beta_j^{-1}(b_i) + \int_{\beta_j^{-1}(b_i)}^{\bar{v}} \beta_j(v)\, dv)$. As in the all-pay auction, if bidder i does not take account of the term κ_i^{F} she would face the usual first-price auction expected payoff.

Lemma 3. *The bidders' equilibrium strategies must be pure strategies that are continuous and strictly increasing functions in v_i.*

Lemma 4. *Minimum and maximum bids must be the same for both bidders so that $\beta_1(0) = \beta_2(0) = 0$, $\beta_1(\bar{v}) = \beta_2(\bar{v}) = \bar{b}$, and $\bar{b} \in [\frac{\bar{v}}{2}, \bar{v})$.*

Lemma 5. *Each bidder submit a nonnegative bid inferior to her value such that $\beta_i(v) < v$ for all $v \in (0, \bar{v}]$ and $i = 1, 2$.*

As in the case of the all-pay auction, from Lemma 3 the inverse function of β_i, ϕ_i, is increasing and differentiable almost everywhere on $[0, \bar{b}]$. Furthermore, $\phi_1(0) = \phi_2(0) = 0$ and $\phi_1(\bar{b}) = \phi_2(\bar{b}) = \bar{v}$. Bidders could not submit a maximum bid higher than their valuation. Furthermore, the maximum bid is bounded because of the limit on the bidders' altruism. The maximum bid in the all-pay auction is therefore higher than the one in the first-price auction.[13]

To derive the equilibrium, as above we state only the necessary condition, whereas the sufficient condition is similar to the one in Theorem 1 of de Frutos (2000) and therefore omitted. Differentiating (9) with respect to b_i it follows

$$\phi_1'(b) = \frac{1 - a_2}{\phi_2(b) - b}\phi_1(b) \quad \text{for all } b \in (0, \bar{b}], \tag{10}$$

$$\phi_2'(b) = \frac{1 - a_1}{\phi_1(b) - b}\phi_2(b) \quad \text{for all } b \in (0, \bar{b}]. \tag{11}$$

There is no explicit solution to this differential equation systems with our boundary conditions. Equations (10) and (11) and the boundary conditions define equilibrium strategies if they define the optimal decision for each bidder.

Proposition 2. *The unique Bayesian Nash equilibrium (β_1, β_2) is characterized by the inverse bidding functions (ϕ_1, ϕ_2) such that*

$$\phi_i(b) = (1 - a_i)\frac{\phi_j(b)}{\phi_j'(b)} + b \quad \text{for all } b \in [0, \bar{b}],$$

which satisfies the boundary conditions $\phi_i(0) = 0$, $\phi_i(\bar{b}) = \bar{v}$, for $i = 1, 2$ and $i \neq j$.

[13]This result is not obvious as for some values of the altruism parameters the maximum bid in the all-pay auction is <1. Claim 1 establishes this result in Section 5.

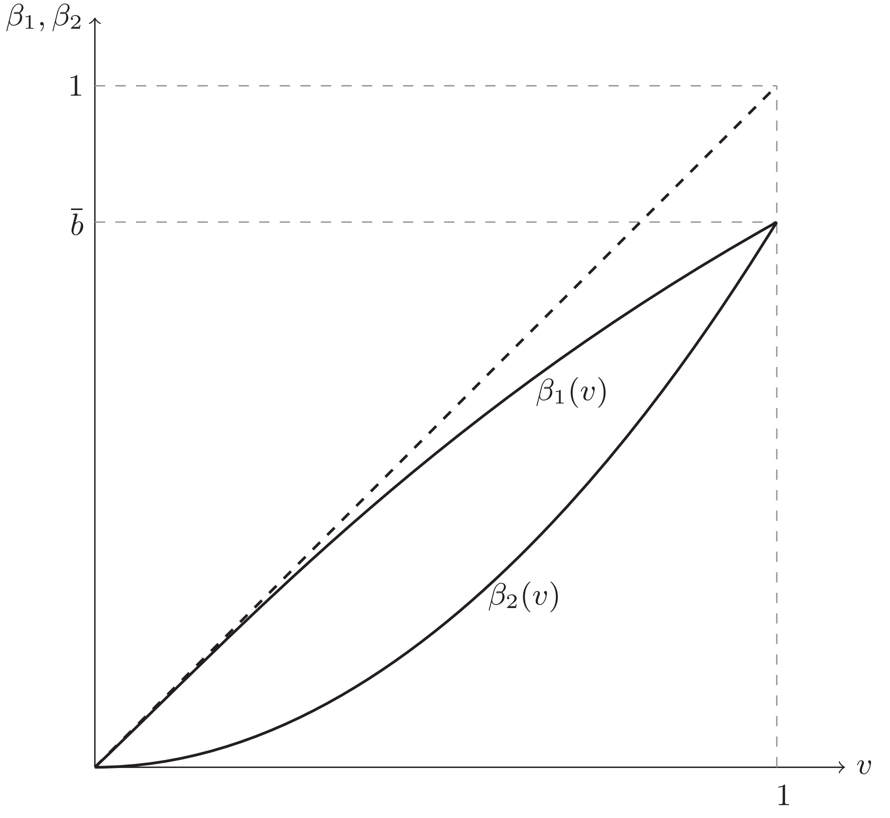

FIGURE 2 Equilibrium bidding strategies for the uniform distribution on [0, 1]

For $a_1 = a_2 \equiv a$ we get the symmetric Nash equilibrium (see Engers & McManus, 2007 for details) such that $\beta_i(v) = \frac{v}{2-a}$ for $i = 1, 2$. The maximum bids, and therefore the expected revenue, are bounded. As in the all-pay auction we can established a strict ranking of the bidding functions.

Corollary 2. *In the first-price auction, the more altruistic bidder is the more aggressive one. More precisely, if $a_1 > a_2$, then $\beta_1(v) > \beta_2(v)$ for all $v \in (0, \bar{v})$.*

This result is useful to determine the shape of the bidding strategies at the equilibrium. Indeed, β_1 and β_2 cannot intersect. Moreover, the equilibrium bidding strategies are concave for Bidder 1 and convex for Bidder 2.[14] Figure 2 depicts the curves of β_1 and β_2 for $\bar{v} = 1$.

5 | REVENUE COMPARISONS

In this section we examine the performance of the all-pay and first-price auctions in terms of the expected revenue. Our next result describes the ranking of the equilibrium bidding strategies for each bidder.

[14]A proof is provided in Section 5, Claim 3.

Lemma 6. *Bidders' i bidding strategies in the all-pay and the first-price auction intersect only once such that*

$$\beta_i(v) \geq \alpha_i(v) \quad \text{for all } v \in [0, \bar{v}_i] \quad \text{and} \quad \alpha_i(v) > \beta_i(v) \quad \text{for all } v \in (\bar{v}_i, \bar{v}], \quad \text{for } i = 1, 2 \quad \text{and} \quad i \neq j.$$

Let us denote e_i^{A} and e_i^{F} the expected payment of bidder i in the all-pay and first-price auctions. These expected payments are $e_i^{\mathrm{A}}(v) = \alpha_i(v)$ and $e_i^{\mathrm{F}}(v) = \frac{1}{\bar{v}}\phi_j(\beta_i(v))\beta_i(v)$ for all $v \in [0, \bar{v}]$.[15] Comparing the expected payments will be useful for ranking the expected revenues.

Proposition 3. *The expected payment of bidder i in the all-pay auction is greater than her expected payment in the first-price auction when her valuation is sufficiently high. Moreover, her expected payment is the same in both auctions when her valuation is equal to zero.*

It is not clear if the expected payment of bidder i in the all-pay auction is greater than in the first-price auction. Indeed it could happen that for some range of valuations the latter outperforms the former. The next proposition determines the ranking of the expected revenue.

Proposition 4. *The expected revenue in the all-pay auction is strictly higher than in the first-price auction.*

Thus, the introduction of the asymmetry on the altruism parameters does not change the ranking of the expected revenue (Engers & McManus, 2007; Goeree et al., 2005). This result was not predictable as the asymmetry can reverse the ranking of the expected revenue in first and second-price auctions (Bulow et al., 1999). Furthermore, this contradicts results with complete information which determine that the first-price auction leads to a higher revenue than the all-pay-auction when bidders are asymmetric enough. Bos (2016) established this revenue comparison for asymmetric valuations. Yet, the ranking of revenues in Bos (2016) also holds for asymmetric altruism parameters, using the Nash equilibrium in mixed strategies for the all-pay auction with asymmetric altruism parameters determined in Bos (2016) and the result of Ettinger (2010) for the first-price auction. Therefore, our result tends to confirm the dominance of the all-pay auction at raising money for charity in an incomplete information framework.

Our results are established for uniform distributions and two bidders, and explore neither all possible probability distributions nor the effect of the number of bidders. However we determine in a setting with uniformly distributed values, independently of the level of asymmetry on the altruism parameters, that the all-pay auction performs better as in the symmetric case with incomplete information and contrary to the asymmetric case with complete information. Moreover, the number of bidders does not have consequences on the ranking of revenues in these both complete and incomplete information settings.

Bidders are aware that a strong asymmetry in the altruism parameters leads to a more aggressive behavior of the bidder with a high level of altruism. In the all-pay auction, that provides incentives, for

[15] Indeed, $e_i^{\mathrm{F}}(v) = \mathbb{P}(\beta_j(V_j) \leq \beta_i(v))\beta_i(v) = \frac{1}{\bar{v}}\phi_j(\beta_i(v))\beta_i(v)$.

the bidder with a low altruism parameter, to reduce her bid. However, despite bidders know the altruism parameters, and then the level of asymmetry, the bidders' values are drawn by a uniform distribution. This uncertainty on the other's value compensates the effect of asymmetry as each bidder can compute her positive probability of winning. There is a positive chance that the other bidder has a sufficiently low type to lose the auction despite a higher altruism parameter. The effect of asymmetry is then balanced enough such that the all-pay auction, and its benefits from all bids paid, raise more money for charity than the first-price auction.

Moreover, the expected revenue in the all-pay auction is given by

$$\mathbb{E}R^{A}(a_1, a_2) = \bar{v}^{\frac{1-a_1}{1-a_2}} \int_0^{\bar{v}} \frac{\alpha_1(v)}{\bar{v}}\,dv + \bar{v}^{\frac{1-a_2}{1-a_1}} \int_0^{\bar{v}} \frac{\alpha_2(v)}{\bar{v}}\,dv$$
$$= \frac{\bar{v}}{2-a_1-a_2}\left(\frac{1-a_2}{3-a_1-2a_2} + \frac{1-a_1}{3-2a_1-a_2}\right).$$

It is interesting to see how the asymmetry affects the expected revenue in the all-pay auction. Consider the altruism parameters are no longer strictly order such that a_1 could be inferior as well as superior than a_2. Let us denote $\bar{a} = a_1 + a_2$, such as $\bar{a} \in [0, 2)$. Upon substitution, we can see that $\mathbb{E}R^{A}(a_1, \bar{a} - a_1)$ is maximized at $a_1 = \frac{\bar{a}}{2}$ and then increasing for $a_1 < \frac{\bar{a}}{2}$ and decreasing for $a_1 > \frac{\bar{a}}{2}$. For example, Figure 3 depicted the situation in which $\bar{a} = 1$ and $\bar{v} = 1$. The next results follow.

Lemma 7. *The greater the asymmetry in the altruism parameters the lower the expected revenue will be in the all-pay auction.*

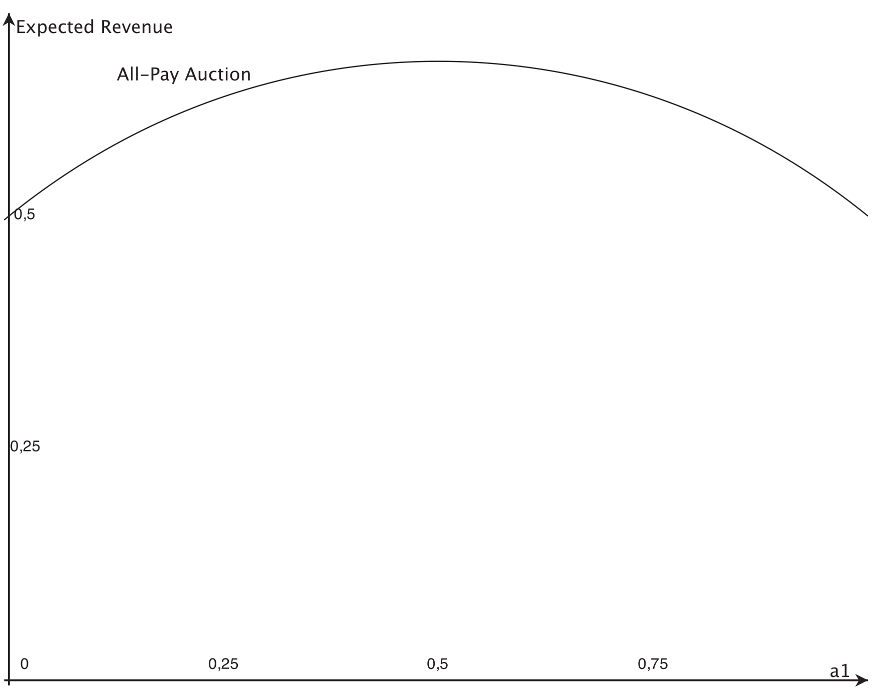

FIGURE 3 Expected revenue of the all-pay auction for $\bar{a} = 1$ and $\bar{v} = 1$

This result is in line with results on asymmetric all-pay auctions with complete information. Hillman and Riley (1989) determine that the expected revenue decreases when the bidders become more asymmetric.

6 | CONCLUSION

The purpose of this paper is to determine which of the two auction designs—the all-pay auction and the first-price auction—is better at raising money for charity when bidders are asymmetric in their altruism parameters with complete information and bidders' valuations are uniformly distributed in an independent private value's model. As in the case with symmetric bidders with incomplete information (Goeree et al., 2005) we conclude that the all-pay auction is better than the first-price auction. This result shows that different auction designs are better for different environments. Indeed, in a complete information framework first-price auctions outperform all-pay auctions when the asymmetry among bidders is strong enough. Moreover, Carpenter et al. (2010) conclude there is no strict ranking of revenues when the participation is endogenous.

Our result also complements the normative approach of Lu (2012), who determines a revenue-maximizing all-pay mechanism, in which all losers pay a fraction of the winner payment and the seller can cancel the auction through a nonparticipation threat. The all-pay auction considered in our analysis should be easier to implement on the field and does not require a nonparticipation threat which is not observed in practice.

This paper, and more generally the idea that the optimal auction design for charity depends on the informational setting, is a good candidate for a lab experiment. Another natural follow-up question concerns dynamic charity auctions. Although Engers and McManus (2007) determined that the second-price and the English auctions are still strategically equivalent, there is no result about the other usual mechanisms, such as the Dutch auction. Recent works about optimal release information in environments with no externality might be useful to investigate this question.[16] Finally, another interesting investigation would be to understand if repeated charity auctions can lead to some charitable cooperations among bidders.[17]

ACKNOWLEDGMENTS

A part of this study was done while I was visiting CORE (Université catholique de Louvain). A Marie-Curie Fellowship from the European Commission and a grant from the European Science Foundation Activity entitled "Public Goods, Public Projects, Externalities" are gratefully acknowledged. This study has also been conducted as part of the Labex MME-DII (ANR11-LBX-0023-01) project. I am grateful to an anonymous associate editor and two anonymous referees for very helpful comments. I am indebted to Claude d'Aspremont and Cédric Wasser whose comments improved the quality of this study. I would also like to thank Gabrielle Demange, Thibault Fally, Martin Hellwig, Paul Heidhues, Konrad Mierendorff, Florian Morath, and Nora Szech for helpful conversations, and the participants of the EEA Conference, the Jamboree EDP Meeting at the LSE, the SAET Conference and seminars at Max Planck Institute (Bonn), Toulouse School of Economics, University of Amsterdam, University of Paris 1, Panthéon-Assas University, and University of St. Andrews for valuable comments. All remaining errors are mine.

[16]For example, see Coleff and Garcia (2017).

[17]See Andreoni, Kuhn, and Samuelson (2019) for a recent experiment about cooperation in repeated games with incomplete information.

ORCID

Olivier Bos http://orcid.org/0000-0003-3148-4207

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How to cite this article: Bos O. Charitable asymmetric bidders. *J Public Econ Theory*. 2020;22:320–337. https://doi.org/10.1111/jpet.12406

APPENDIX

Proof of Lemma 1. Lemma 1 in de Frutos (2000) determines the same result for the first-price auction with asymmetric values and symmetric altruism parameters equal to 1/2. Our result, for the all-pay auction with asymmetric altruism parameters, can be established with similar technical arguments. Therefore, the proof is omitted and available on request. □

Proof of Lemma 2. Assume that $0 \leq \alpha_i(0) \leq \alpha_j(0)$. Each bidder gets the same payoff by winning as well as losing. As bidders have a strict preference for a higher payoff independently of the outcome, it follows that $\alpha_i(0) = \alpha_j(0) = 0$. Assume that $\alpha_j(\bar{v}) > \alpha_i(\bar{v})$. Then, the Bidder 1 can decrease her bid without affecting her winning probability and increasing her payoffs. Similarly, $\alpha_i(\bar{v}) > \alpha_j(\bar{v})$ cannot be part of the equilibrium. Thus, $\alpha_1(\bar{v}) = \alpha_2(\bar{v})$. □

Proof of Proposition 1. It is clear that at the equilibrium $\alpha_i(0) = 0$. Indeed, if $b_i = 0$ the payoff of the bidder i for $v_i > 0$ is strictly inferior to the one for $v_i = 0$. Consider now the payoff of the bidder i for all $b_i \in (0, \bar{b}]$.

$$\begin{aligned}\frac{\partial U_i^{\mathrm{A}}}{\partial b_i}(v_i, b_i, \alpha_j; a_i) &= \frac{v_i}{\bar{v}}\varphi_j'(b_i) - (1 - a_i)\\ &= \frac{v_i - \varphi_i(b_i)}{\bar{v}}\varphi_j'(b_i).\end{aligned}$$

To get the last line we used the necessary condition $\varphi_i(b_i)\varphi_j'(b_i) = \bar{v}(1 - a_i)$. When $v_i > \varphi_i(b_i)$ it follows that $\frac{\partial U_i^{\mathrm{A}}}{\partial b_i}(v_i, b_i, \alpha_j; a_i) > 0$. In a similar manner, when $v_i < \varphi_i(b_i)$, $\frac{\partial U_i^{\mathrm{A}}}{\partial b_i}(v_i, b_i, \alpha_j; a_i) < 0$. Thus, $\frac{\partial U_i^{\mathrm{A}}}{\partial b_i}(v_i, \alpha_i, \alpha_j; a_i) = 0$. As a result, the maximum of $U_i^{\mathrm{A}}(v_i, \alpha_i, \alpha_j; a_i)$ is achieved for $v_i = \varphi_i(b_i)$ and then $b_i = \alpha_i(v_i)$. □

Proof of Corollary 1. Recall that we assume $a_1 > a_2$. The result follows. □

Proof of Lemma 3. This proof can be established with the same technical arguments than Lemma 1 in de Frutos (2000) and therefore omitted. □

Proof of Lemma 4. and 5. Assume that $\beta_i(0) < \beta_j(0)$. When the valuation is 0, the payoff of losing is higher than the payoff of winning. Then, both bidders deviate and submit a bid equal to 0 such that $\beta_1(0) = \beta_2(0) = 0$.

Using the same mathematical arguments than in Lemma 1 of de Frutos (2000), we can be established $\bar{b} < \bar{v}$. Let us show that $\bar{b} \geq \frac{\bar{v}}{2}$. Summing differential equations (10) and (11) it follows

$$\phi_1'(b)\phi_2(b) + \phi_2'(b)\phi_1(b) - b(\phi_1'(b) + \phi_2'(b)) - (\phi_1(b) + \phi_2(b))$$
$$= -a_1\phi_2(b) - a_2\phi_1(b).$$

Integrating this equation and using $\phi_i(\bar{b}) = \bar{v}$,

$$\bar{v}(\bar{v} - 2\bar{b}) = -\int_0^{\bar{b}} a_1\phi_2(x) + a_2\phi_1(x)\,dx.$$

Hence the result. □

Proof of Proposition 2. The existence and uniqueness follows from Theorem 2 and Corollary 4 in Lebrun (1999), as in Theorem 3 of de Frutos (2000). □

Proof of Corollary 2. This proof uses the same technical arguments than Proposition 4.4 in Krishna (2009) and therefore omitted. □

Proof of Lemma 6. To prove this result we first establish properties of the bidding strategies. □

Claim 1. The maximum bid in all-pay auction is higher than that in first-price auction for nonnegative altruism parameters.

Proof. Let us denote $\bar{b}^{\mathrm{A}}$ and $\bar{b}^{\mathrm{F}}$ the maximum bids in the all-pay and first-price auctions, respectively. Clearly, $\bar{b}^{\mathrm{A}} \geq \bar{v} > \bar{b}^{\mathrm{F}}$ for all $a_1 + a_2 \geq 1$. Let us assume that $\bar{b}^{\mathrm{F}} \geq \bar{b}^{\mathrm{A}}$ for some $a_1 + a_2 < 1$. Then, by continuity there exists a value of $a_1 + a_2$ such that $\bar{b}^{\mathrm{F}} = \bar{b}^{\mathrm{A}}$. If this case happens with asymmetric bidders that also happens with symmetric bidders. In the latter case, $a_1 + a_2 = a$, $\bar{b}^{\mathrm{F}} = \frac{\bar{v}}{2-a}$, and $\bar{b}^{\mathrm{A}} = \frac{\bar{v}}{2(1-a)}$. Hence the result. □

Claim 2. $\varphi_i(b) > \phi_i(b)$ and $\varphi_j(b) > \phi_j(b)$ for all b close to 0.

Proof. Using L'Hôspital's rule in (10) implies

$$1 - a_i = \lim_{b\to 0}\phi_j'(b)\frac{\phi_i(b) - b}{\phi_j(b)} = \phi_j'(0)\lim_{b\to 0}\frac{\phi_i(b) - b}{\phi_j(b)}$$
$$= \phi_j'(0)\lim_{b\to 0}\frac{\phi_i'(b) - 1}{\phi_j'(b)} = \phi_i'(0) - 1.$$

Thus, $\phi_i'(0) = 2 - a_i$ for $i = 1, 2$.

As $\varphi_i'(b) = \bar{v}^{\frac{1-a_i}{2-a_i-a_j}}(1 - a_j)((2 - a_i - a_j)b)^{\frac{-1+a_i}{2-a_i-a_j}}$, and $a_i > a_j$, $\lim_{b\to 0}\varphi_i'(b) = +\infty$. Hence, $\varphi_i'(0) > \phi_i'(0)$ for $i = 1, 2$. Therefore, $\varphi_i(b) > \phi_i(b)$ for all b sufficiently close to 0 and $\beta_i(v) > \alpha_i(v)$ for all v sufficiently close to 0. □

Claim 3. The inverse bidding strategies ϕ_1 and ϕ_2 are, respectively, convex and concave functions.

Proof. Remark that from (10) and (11) ϕ_1 and ϕ_2 are continuous functions and therefore differentiable. From (10) and (11) we obtain

$$\phi_i''(b) = \frac{1-a_j}{(\phi_j(b)-b)^2}(\phi_i'(b)(\phi_j(b)-b)-(\phi_i(b)(\phi_j'(b)-1))) \quad \text{for } i=1,2 \text{ and } i\neq j. \tag{A1}$$

Let us assume that $\phi_2''(b)>0$ for all $b\in[0,\bar{b}^{\mathrm{F}}]$. Note that $\phi_1''(b)<0$ is equivalent to $\frac{\phi_1'(b)}{\phi_1(b)}<\frac{\phi_2'(b)-1}{\phi_2(b)-b}$. Using (10), this is also equivalent to $\phi_2'(b)>2-a_2$. Thus, as $\phi_2'(0)=2-a_2\phi_2$ convex leads to ϕ_1 concave. Yet, ϕ_1 concave, ϕ_2 convex, and the boundary conditions contradict Corollary 2. Hence, ϕ_2 cannot be convex.

Let us assume that ϕ_2 is neither convex nor concave. Then there exists at least one inflection point b, such as $\phi_2''(b)=0$. Denote $\tilde{b}$ the first inflection point. Then, $\phi_2''(\tilde{b})=0$ and (A1) imply $\phi_1'(\tilde{b})=2-a_1$. As $\phi_1'(0)=2-a_1$, ϕ_1' is not strictly monotone on $[0,\tilde{b}]$ and there exists $\tilde{\tilde{b}}$, such as $\phi_1''(\tilde{\tilde{b}})=0$ with $\tilde{\tilde{b}}<\tilde{b}$.[18] In the same way, $\phi_1''(\tilde{\tilde{b}})=0$ and (A1) imply $\phi_2'(\tilde{\tilde{b}})=2-a_2$. As $\phi_2'(0)=2-a_2$, ϕ_2' is not monotone on $[0,\tilde{\tilde{b}}]$ which contradicts that $\tilde{b}$ is the first inflection point of ϕ_2.[19] Hence, ϕ_2 has to be either convex or concave. With a symmetric argument we get the same result for ϕ_1.

In consequence $\phi_2''(b)\leq 0$ for all $b\in[0,\bar{b}^{\mathrm{F}}]$. Furthermore, $\phi_1''(b)\geq 0$ if and only if $2-a_2\geq\phi_2'(b)$ which is true as ϕ_2 is concave and $\phi_2'(0)=2-a_2$. Hence, ϕ_1 is convex. □

Claim 4. The inverse bidding strategy φ_i is a concave function.

Proof. Differentiating twice (8) leads to $\varphi_i''(b)=-\bar{v}^{\frac{1-a_i}{2-a_i-a_j}}(1-a_j)(1-a_i)((2-a_i-a_j)b)^{\frac{-3+2a_i+a_j}{2-a_i-a_j}}$ for all $b\in[0,\bar{b}^{\mathrm{A}}]$, which is negative. □

Claims 2–4 imply that the curves ϕ_i and φ_i intersect once and only once. Moreover, $\varphi_i(b)\geq\phi_i(b)$ for all $b\in[0,\tilde{b}_i]$ with $\tilde{b}_i<\bar{b}^{\mathrm{F}}$ and $\varphi_i(b)<\phi_i(b)$ for all $b\in[\tilde{b}_i,\bar{b}^{\mathrm{F}}]$. Furthermore, we have shown that $\alpha_i(v)>\beta_i(v)$ for all $v\in[\bar{v}_i,\bar{v}]$ with $\alpha_i(\bar{v}_i)=\bar{b}^{\mathrm{F}}$. This completes the proof of Lemma 6. □

Proof of Proposition 3. The expected payment of Bidder 1 from the first-price auction is given by $e_1^{\mathrm{F}}(v)=\frac{1}{\bar{v}}\phi_2(\beta_1(v))\beta_1(v)$. Then, $e_1^{\mathrm{F}}(0)=0$ and $e_1^{\mathrm{F}}(\bar{v})=\bar{b}^{\mathrm{F}}$. As β_1 and ϕ_2 are both positive and increasing functions, e_1^{F} is also a positive and increasing function. Moreover $e_1^{\mathrm{F}}(0)=\alpha_1(0)=0$ and $e_1^{\mathrm{F}}(\bar{v})<\alpha_1(\bar{v})$. Therefore, the expected payments are equal in both auctions when the Bidder 1's valuation is equal to zero. Moreover, the Bidder 1's expected payment in the all-pay auction is greater than her expected payment in the first-price auction when her valuation is sufficiently high. As $e_2^{\mathrm{F}}(0)=\alpha_2(0)=0$ and $e_2^{\mathrm{F}}(\bar{v})<\alpha_2(\bar{v})$, the same result holds for the Bidder 2. □

[18] If ϕ_1' is constant on $[0,\tilde{b}]$, ϕ_2' is also constant on this interval and $\tilde{b}$ cannot be an inflection point.

[19] If ϕ_2' is constant on $[0,\tilde{\tilde{b}}]$, ϕ_1' is also constant on this interval. Thus, $\tilde{\tilde{b}}$ cannot be an inflection point for ϕ_1.

Proof of Proposition 4. Before showing the result, let us establish inequality (A2).

Claim 5.

$$\int_0^{\bar{v}} \frac{x^2}{2\bar{v}^2}\beta_i'(x)\,dx = \bar{v}\int_0^1 \frac{x^2}{2}\beta_i'(\bar{v}x)\,dx \geq \bar{v}\int_0^1 \frac{x^2}{2}dx\int_0^1 \beta_i'(\bar{v}x)\,dx$$
$$= \int_0^{\bar{v}} \frac{x^2}{2\bar{v}^2}dx\int_0^{\bar{v}} \beta_i'(x)\frac{1}{\bar{v}}dx \quad \text{for } i = 1, 2. \tag{A2}$$

Proof. The left and right terms of the inequality (A2) are established thanks to an integration by parts. β_2' is an increasing function. Then, for $i = 2$ (A2) is a special case of the Chebyshev's inequality for monotone functions. Yet, this inequality cannot be applied for $i = 1$ as β_1' is decreasing. However, $\int_0^{\bar{v}} \frac{x^2}{2\bar{v}^2}\beta_1'(x)\,dx \geq \int_0^{\bar{v}} \frac{x^2}{2\bar{v}^2}dx\int_0^{\bar{v}} \beta_1'(x)\frac{1}{\bar{v}}dx$ is equivalent to $\int_0^1 \frac{x^2}{2\bar{v}^2}(\beta_1'(x) - \frac{\bar{b}^{\mathrm{F}}}{\bar{v}})dx \geq 0$. Then, let us show that $\beta_1'(x) \geq \frac{\bar{b}^{\mathrm{F}}}{\bar{v}}$ for all $x \in [0, \bar{v}]$. Notice, $\beta_1'(x) \geq \beta_1'(\bar{v})$ and $\bar{b}^{\mathrm{F}} \leq \frac{\bar{v}}{2-a_2}$ as β_1' is decreasing and the maximum bid with asymmetric bidders cannot be higher than the maximum bid with symmetric bidders. Therefore, it is enough to establish that $\beta_1'(\bar{v}) \geq \frac{1}{2-a_2}$. Suppose the contrary which is equivalent to $\phi_1(\bar{b}^{\mathrm{F}})' > 2 - a_2$. This inequality is also equivalent to $\frac{1-a_1}{1-\bar{b}^{\mathrm{F}}}\bar{v} > 2 - a_2$ which leads to $\bar{b}^{\mathrm{F}} > \frac{\bar{v}}{2-a_2}$; hence a contradiction. □

Denote by Δ_i the difference among $\frac{1}{\bar{v}}\int_0^{\bar{v}} e_i^{\mathrm{A}}(v)\,dv$ and $\frac{1}{\bar{v}}\int_0^{\bar{v}} e_i^{\mathrm{F}}(v)\,dv$. Then for $i = 1, 2$ and $i \neq j$,

$$\Delta_i = \frac{1}{\bar{v}}\int_0^{\bar{v}} \left(\alpha_i(v) - \frac{1}{\bar{v}}\phi_j(\beta_i(v))\beta_i(v)\right) dv \tag{A3}$$

$$\geq \bar{b}^{\mathrm{A}} - \frac{1}{\bar{v}}\int_0^{\bar{v}} v\alpha_i'(v)\,dv - \frac{\bar{b}^{\mathrm{F}}}{2} + \int_0^{\bar{v}} \frac{v^2}{2\bar{v}^2}\beta_i'(v)\,dv \tag{A4}$$

$$= \bar{b}^{\mathrm{A}} - \frac{1}{\bar{v}}\int_0^{\bar{v}} v\alpha_i'(v)\,dv - \frac{\bar{b}^{\mathrm{F}}}{3} \tag{A5}$$

$$\geq \frac{\bar{v}}{2 - a_1 - a_2} - \frac{\bar{v}}{3 - 2a_j - a_i} - \frac{\bar{v}}{3(2 - a_2)}. \tag{A6}$$

Using $v \geq \phi_1(\beta_2(v))$ from Corollary 2 and integrating by parts (A4) follows. Using Claim 5 we obtain (A5). To get (A6) remark that the maximum bid with asymmetric bidders cannot be higher than the maximum bid with symmetric bidders. Then it follows

$$\Delta_1 \geq \bar{v}\frac{5a_1 - a_1^2 - 3a_1a_2 - 2a_2 + a_2^2}{3(2 - a_2)(2 - a_1 - a_2)(3 - a_1 - 2a_2)},$$
$$\Delta_2 \geq \bar{v}\frac{a_1 - 2a_1^2 + 2a_2 - a_2^2}{3(2 - a_2)(2 - a_1 - a_2)(3 - 2a_1 - a_2)},$$

and $\Delta_1 + \Delta_2 \geq \frac{\bar{v}\delta(a_1, a_2)}{3(2 - a_2)(2 - a_1 - a_2)(3 - 2a_1 - a_2)(3 - a_1 - 2a_2)}$ with

$$\delta(a_1, a_2) = (3 - a_1 - 2a_2)(a_1 - 2a_1^2 + 2a_2 - a_2^2) + (3 - 2a_1 - a_2)(5a_1 - a_1^2 - 3a_1a_2 - 2a_2 + a_2^2).$$

Let us show that the function $\delta(a_1, a_2)$ is positive for all a_1 given a_2 fixed and $a_1 > a_2$.

First, note that for each value of a_2 inferior to a_1:

- $\delta(a_1, a_2) \underset{a_1 \to a_2}{\longrightarrow} 18(-1 + a_2)^2 a_2 > 0$.
- $\delta(a_1, a_2) \underset{a_1 \to 1}{\longrightarrow} 2 - 3a_2 + a_2^3 > 0$.

Moreover, $\frac{\partial \delta}{\partial a_1}(a_1, a_2) = 2[6a_1^2 + a_1(11a_2 - 20) + 9 - 7a_2 + a_2^2]$. Then, to determine the monotonicity of δ given a_2 requires the determination of the sign of the polynomial

$$6a_1^2 + a_1(11a_2 - 20) + 9 - 7a_2 + a_2^2. \tag{A7}$$

The discriminant of Equation (A7) is $85a_2^2 - 188a_2 + 76$ and thus nonpositive for all $a_2 > \underline{a}_2 \equiv \frac{94 - 2\sqrt{594}}{85} \sim 0.532$. Therefore, for all $a_1 \in (\underline{a}_2, 1)$ given $a_2 > \underline{a}_2$ the function δ is increasing in a_1. Hence, $\Delta_1 + \Delta_2 > 0$.

Yet, when $a_2 \leq \underline{a}_2$ Equation (A7) could positive as well as negative. Indeed, (A7) is positive for all $a_1 \leq \underline{a}_1$ and nonpositive for all $a_1 > \underline{a}_1$ with $\underline{a}_1 \equiv \frac{20 - 11a_2 + \sqrt{85a_2^2 - 188a_2 + 76}}{12}$. Note that $\underline{a}_1$ is positive but superior to 1 when $a_2 > \tilde{a}_2 \equiv \frac{-1 + \sqrt{13}}{6} \sim 0.4342$. Then, we have to distinguish two cases:

- For all $a_1 \in (0, 1)$ given $a_2 < \tilde{a}_2$, δ is increasing for $a_1 \in (0, \underline{a}_1]$ and decreasing for $a_1 \in [\underline{a}_1, 1)$. It follows that $\Delta_1 + \Delta_2 > 0$.
- For all $a_1 \in [\tilde{a}_2, 1)$, such as $a_2 \in [\tilde{a}_2, \underline{a}_2]$, δ is increasing. Hence, $\Delta_1 + \Delta_2 > 0$.

Finally, we have determined that the function δ is nonnegative for all a_1 given each value of a_2 inferior to a_1. This completes the proof. □