Existence of Equilibrium in All-Pay Auctions with Price Externalities*

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Abstract

This paper investigates all-pay auctions with general price externalities and complete information. We show the existence of a mixed strategy Nash equilibrium by using Schauder's fixed-point theorem. The Brouwer's fixed-point cannot be applied because of the infinite-dimensional set of distribution functions. Our findings are applicable to future works on contests and charity auctions.

Keywords: All-pay auction, contest, externalities

JEL Classification: C72, D44, D62

1 Introduction

All-pay auctions are usually studied either as an auction or a contest. In this game, all bidders have to pay their bids and the one who submitted the highest bid wins. Fundraising mechanisms, race competitions and R&D investments are three of the many applications.[1] Competitors in a race care about the recognition they could get from their participation. Therefore, a loser is better off with an effort closer to the winner's effort and a winner with an effort further to the highest loser's effort. In a charity auction, bidders usually care about the charity purpose and benefit from the total amount raised during the auction. In both situations, participants benefit from a price externality, either dependent or independent of the winner identity.[2]

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[1]An investment game is discussed pp. 2-3. In that case, solely the winner is bid-dependent payoff.

[2]Notice there is also a large literature about auctions with allocative externalities initiated by Jehiel and Moldovanu (1996) with complete information. The key in this literature is the winner identity and not, as here, the money/efforts spent by all opponents. See Klose and Kovenock (2015) for an analyzis of the all-pay auction with allocative externalities. Following Ettinger (2010), we use the terminology *price externality* to distinguish the concern of others' payment/effort to allocative externalities.

The most famous result in this literature is that all-pay auctions are optimal mechanisms at raising money for charity (Goeree et al. (2005) and Engers and McManus (2007)). However, this is not confirmed on the field. Carpenter et al. (2008) and Onderstal et al. (2013) observe a low participation in field experiments and determine other better fundraising mechanisms. This might be explained by the assumption of linear externality that participants benefit, on which rely all theoretical results in this literature.[3] Understanding the role plays by non-linear preferences, in particular for the all-pay auction, could bring substantial results about charity auctions. First, it might likely explain the puzzle observed on the field. Second, it could open new perspectives to better understand the role of the all-pay auction in the charitable mechanisms and other common used mechanisms such as lotteries. Finally, that would lead to determine optimal mechanisms and provide better policy implications. This paper is a step forward to this direction.

We propose to investigate the all-pay auction with complete information without specifying the shape of the externality functions. Complete information is not as usual in auction theory as in the contest literature. However, there are recent works about auctions with externalities which investigate a setting with complete information.[4]

Our analysis does not make any assumptions on the analytic form of the externality function, and consider that bidders take in account a positive externality from their own bid and either a positive or a negative externality from their rivals' bids.[5] Thus, the externality function might also reflect a link between the bids of a bidder and her competitors.[6] These are relevant with economic applications, and more specifically with charity auctions and race competitions. We establish the existence of a Nash equilibrium with mixed strategies, defined on a closed, convex and infinite-dimensional set of continuous distribution functions. Therefore we use the Schauder's fixed-point theorem to determine this result despite the infinite-dimensional set of functions. That might be useful for future research on fundraising mechanisms.

Our paper is also related to the literature about contests with non-monotonic payoffs. They study an all-pay auction with prize value depending of their bids, and provide a novel analysis of R&D races (see for example Chowdhury (2017)). Consider an investment game. Firms undertake expenses to elaborate an innovation and then improve either their own technology or

[3]Among the other possible explanations, Carpenter et al. (2010) suggest the unfamiliarity of the participants to the mechanism and endogenous participation (which is not considered here) and Bos (2016) heterogeneity among bidders.

[4]Among them, Jehiel and Moldovanu (1996) investigate the consequences of allocative externalities on participation decisions in the first-price winner-pay auction, Konrad (2006) examines ownership structure through an all-pay auction with externalities dependent of the firms' identity, Ettinger (2010) investigates the first-price and second-price winner-pay auctions with price externalities, Klose and Kovenock (2015) analyze the all-pay auction with allocative externalities, Bos (2016) compares the performance of all-pay auctions and winner-pay auctions for charity purpose and Damianov and Peeters (2018) compare the lowest-price all-pay auction with other fundraising mechanisms.

[5]Such situations are for example charity auctions, in which bidders benefit from the revenue raised, and therefore all bids including their own. In a contest, a participant with a positive effort can also be perceived positively regarding her own effort, and derogatory regarding her opponents' effort strength.

[6]This could be for example a form of correlation.

their product, before the competitive market (for example in a Bertrand competition). Therefore, all firms must support their expenses, corresponding to a sunk cost. The firm with the best innovation, the highest level of investment, will then get the largest market shares, and evaluate them regarding her corresponding expenses. The essential difference features with our analysis are twofold. Contrary to our analysis, the corresponding bid-dependent payoffs is relevant only for the winner, who does not take in account an externality from her opponents' bids. As a consequence, it exists a Nash equilibrium in pure strategies (Amegashie, 2001, Sacco and Schumtzler, 2008, Siegel, 2014, Chowdhury, 2017). Our approach can be used to establish the existence a Nash equilibrium with mixed strategies when the bid supported is linearly separable in the prize value (Amegashie, 2001, Sacco and Schumtzler, 2008, Bos and Ranger, 2018).

The remainder of this paper is structured as follow. Section 2 introduces the formal setting and properties of general price externalities. In Section 3 we discuss the non-existence of a pure strategy Nash equilibrium and show the existence of a mixed strategy Nash equilibrium. Section 4 concludes. In the following we refer to the players as *bidders*, keeping the auction terminology.

2 The Model

Suppose n risk-neutral bidders submit their bids for an indivisible object (or prize) which is allocated to the highest bidder. We denote the set of the bidders by $\mathcal{N} = \{1, ..., n\}$. Bidder i's value is given by v_i such that $v_1 \geqslant v_2 \geqslant ... \geqslant v_n \geqslant 0$. Although valuations are common knowledge among the potential bidders, the seller has no information about them. All bidders have to pay their bids. Therefore, either bidder i wins the auction with a bid x_i, and obtains a payoff $v_i - x_i$, or she loses and obtains $-x_i$.[7] Moreover, the money raised from each potential bidder potentially impacts the utility of each other: each bidder benefits from her own participation and either benefits or suffers from her rivals' bids.

Thus, the bidder's utility function includes a price externality depending on all bids paid. That could be a function of the sum of the bids $\sum_{i=j}^{n} x_j$, the revenue raised, as in charity auctions, and thus independent of the winner's identity. That could also be a function of the difference between bids contingent on the winner's identity. Indeed in contests, participants might get a reward not only from winning but also from her relative performance compared to others. Accordingly we consider an externality function that depends on all bids,

$$h_i : [0, \bar{x}_i] \times_{j=1, j\neq i}^{n} [0, \bar{x}_j] \to \mathbb{R}_+$$

with $\bar{x}_k$ the maximum bid of bidder k, for all $k \in \mathcal{N}$. It follows that bidder i's utility is given by[8]

$$U_i(x_i, \boldsymbol{x}_{-i}) = \begin{cases} v_i - x_i + h_i(x_i, \boldsymbol{x}_{-i}) & \text{if } i \text{ is the only winner} \\ \frac{v_i}{k} - x_i + h_i(x_i, \boldsymbol{x}_{-i}) & \text{if } i \text{ is one of the } k \text{ winners} \\ -x_i + h_i(x_i, \boldsymbol{x}_{-i}) & \text{otherwise} \end{cases} \quad (1)$$

[7]If more than one bidder submit the same winning bid, they win with equal probability.

[8]Bidder i might be one of the k winners, all submitting the same highest bid, such that $k = \#\{j | j = \arg\max\{x_i, i \in \mathcal{N}\}$.

with $\boldsymbol{x}_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$. The linear case can lead to $\alpha_i \sum_{j=1}^{n} x_j$, with α_i a positive number, which is the usual form of the externality function in the charity auction literature[9] and other works about auctions with identity-independent price externalities[10]. For the analysis we make the following assumptions, relevant with economic applications.

Remark that if the externality function was independent of the opponents' bids, $h_i(x_i, \boldsymbol{x}_{-i}) = h_i(x_i)$, the utility function would correspond to a specific case of bid-dependent prizes.[11] Moreover the following assumption $A3$ does not allow bidders to support a negative externality from their own bids, as in Sacco and Schumtzler (2008).

Assumption 1 ($A1$)**.** *If all bidders submit a zero bid, none benefit from the externality:* $h_i(0, ..., 0) = 0$.

Assumption 2 ($A2$)**.** *The externality function is a* $\mathcal{C}^1$ *function.*

As stated by Assumption $A1$, it is relevant that non-active participation of all bidders does not induce any benefit. Assumption $A2$ rules out discontinuities, which would occur in some specific environments.[12]

Assumption 3 ($A3$)**.** *Bidder* i *benefits from a weakly increasing externality in her own bid over* $[0, \bar{x}_i]$.

Although bidder i gets a benefit from a rise in her own bid, such as $\dfrac{\partial h_i}{\partial x_i}(x_i, \boldsymbol{x}_{-i}) \geqslant 0$, this is not necessarily the case from her rivals' bids. This is contingent on the shape of h_i and therefore how bidder i perceives her competitors' bids and the link between all bids.

Assumption 4 ($A4$)**.** *Bidder* i*'s payoff is weakly decreasing with her payment* $x_i \in [0, \bar{x}_i]$ *in the auction.*

Assumption $A4$ means that conditional of winning or losing, bidders are always better with a lower payment, and leads to $\dfrac{\partial h_i}{\partial x_i}(x_i, \boldsymbol{x}_{-i}) \leqslant 1$. In the literature about about all-pay auctions with price externalities, it is either implicitly or explicitly assumed that the payoff cannot increase with the bidder payment, which means that $\dfrac{\partial h_i}{\partial x_i}(x_i, \boldsymbol{x}_{-i}) < 1$. Applied to charity auctions, this assumption displays the limit of the bidders' altruism. Otherwise bidders could be indifferent between giving and keeping money for their personal use.[13] In a contest, such as a race, a competitor gets a higher (lower) payoff from a better (worst) relative performance, $\alpha_i(x_i - \max_{j \neq i} x_j)$, due to reputation or recognition. Yet the marginal cost has to be higher than the marginal benefit from her own performance, otherwise her effort might go to infinity. That requires again $\alpha_i < 1$, which is here $\dfrac{\partial h_i}{\partial x_i}(x_i, \boldsymbol{x}_{-i}) < 1$. Our investigation is more general. We

[9]See Goeree et al. (2005), Engers and McManus (2007) and Bos (2016).

[10]See for example Ettinger (2010).

[11]See for example the utility form in Amegashie (2001) and Bos and Ranger (2018).

[12]For example, that would be the case if bidders care about rank of bids.

[13]Goeree et al. (2005) and Bos (2016) consider the linear case $\alpha \sum_{i=1}^{n} x_i$ and assume that α is strictly inferior to 1. Engers and McManus (2007) add a *warm glow* a la Andreoni (1989), making bidders more sensitive to their own bid for cognitive/psychological reasons and thus benefiting differently from their own bids and their rivals' bids. Therefore, bidder i benefits from the externality $\alpha x_i + \beta \sum_{j=1, j \neq i}^{n} x_j$ with $1 > \alpha \geqslant \beta > 0$.

also consider the limit case such that bidders could be fully altruistic in a charity auction and the marginal cost of the winner in a race could be equal to her marginal benefit. The next example provides an illustration for a plausible non-linear externality function h.

Example 1 (Non-linear externality)**.** *Consider a contest, such as a race, in which the good (bad) perception of a competitor can be a mixture of her relative performance and the absolute performance of all participants. For example, suppose the winner's relative effort regarding the highest loser's effort is 0.25. She benefits from a better reputation if her absolute effort superior to 1, and a worst reputation if it is equal to 0.5. For the loser with the highest effort, being able to not be far from the highest effort is awarded for small absolute efforts. This means that the strength of the relative effort plays a different role for the winner and the losers. Those effects can be represented by the function*

$$\alpha_i(x_i - (\max_{j \neq i} x_j)^{\frac{1}{2}}) \text{ with } \alpha_i < 1, \text{ for all } i, j \in \mathcal{N}, i \neq j.$$

Note that assumptions A1 to A4 are satisfied.

3 Existence of an Equilibrium

In this section, we show the existence of a Nash equilibrium in mixed strategies. To understand better the extent of this result, we first discuss how the (non) existence of a Nash equilibrium in pure strategies is contingent on the shape of the externality functions.

It is a well known result there is no bidding equilibrium in pure strategies in all-pay auctions without price externality (see Hillman and Riley (1989) and Baye et al. (1996)). Unsurprisingly, that is also the case for the class of externality functions such that $\frac{\partial h_i}{\partial x_i}(x_i, \boldsymbol{x}_{-i}) < 1$ for all $i = 1, ..., n$. A sketch of proof is provided in Appendix.

Therefore, the existence of a bidding equilibrium in pure strategies is not guaranteed and is fundamentally contingent on a specific and favorable shape of the externality functions. Therefore, we are looking for the existence of a Nash equilibrium in mixed strategies. In the following we denote $F_i(x) \equiv \mathbb{P}(X_i \leqslant x)$ the cumulative distribution functions such that bidder i decides to submit a bid lower than x. $F_1, ..., F_n$ can be interpreted as the bidding (mixed) strategies where the support is a strict subset of $\mathbb{R}_+$. Whatever the outcome of the auction, once bidder i computes her expected utility, she takes the bids paid by all bidders into account, including her own. Remark that we do not know if the cumulative distributions $F_1, ..., F_n$ admit density functions. Therefore we use the Stieltjes integral for the expected utility and in the following to determine the existence of a mixed strategy Nash equilibrium.[14] Moreover, the Stieljes integral exists if the cumulative distribution F_i is a function of bounded variation[15] and then is discontinuous in at most a countable set of points. It follows that the cumulative distributions might

[14]See (for example) Carothers (2000, Chap. 14) about the use of the Stieltjes integral because of non-existence of density functions.

[15]A cumulative distribution can be built from the difference between two bounded monotone functions and then be a function of bounded variation.

not have density functions, but can admit atoms and a finite number of discontinuous points as in the standard case with no externalities Baye et al. (1996).

The next Lemma shows that a result determined by Siegel (2009), about atoms in a general form of contests with no externalities, also holds in the all-pay auction with general price externalities.

Lemma 1. *Assume that at least two bidders have an atom at x. Then all bidders with an atom at x lose with probability 1 by submitting x.*

Proof. Consider $\mathcal{T}$, $|\mathcal{T}| \geqslant 2$, the set of bidders who have an atom at x, which is bidders bid x with strictly positive probability. Let us show the event "a winning tie occurs at x" has a zero probability.

Assume this event has a strictly positive probability. The object is allocated among the $|\mathcal{T}|$ winners, such as every $i \in \mathcal{T}$ gets her value divided by the number of winners, i.e., $\frac{v_i}{|\mathcal{T}|}$. Bidder i, by submitting x with a strictly positive probability, must have a positive payoff, $\frac{v_i}{|\mathcal{T}|} - x + h_i(x, \tilde{\boldsymbol{x}}_{-i}) \geqslant 0$ for a given vector of the other bids $\tilde{x}_{-i}$. Then there exists an $\varepsilon > 0$ such as $v_i - (x + \varepsilon) + h_i(x + \varepsilon, \tilde{\boldsymbol{x}}_{-i}) \geqslant \frac{v_i}{|\mathcal{T}|} - x + h_i(x, \tilde{\boldsymbol{x}}_{-i})$, i.e., for which i can increase her probability of winning to 1 and her payoff by submitting $x + \varepsilon$. Therefore, the event "a winning tie occurs at x" has a zero probability to occur. This implies there is at least one bidder in $\mathcal{N} \backslash \mathcal{T}$ who submits a higher bid than x and wins the auction. Hence, all bidders with an atom at x will lose with probability 1 by submitting x.

■

The Lemma 1 implies that ties, in which bidders win with a strictly positive probability, are zero probability events. It follows that bidder i's expected utility is given by,[16]

$$\mathbb{E}U_i(x_i, \boldsymbol{X}_{-i}) = \Pi_{j \neq i} F_j(x_i) v_i - x_i + \mathbb{E}h_i(x_i, \boldsymbol{X}_{-i}) \text{ for all } i = 1, ..., n$$

with $\boldsymbol{X}_{-i} = (X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)$. A potential bidder takes part in the auction if for some positive bids her expected utility is at least equal to the externality she benefits by bidding zero. Formally if

$$\exists x_i \text{ such that } \mathbb{E}U_i(x_i, \boldsymbol{X}_{-i}) \geqslant \mathbb{E}\left(h_i(0, \boldsymbol{X}_{-i})\right)$$

If the externality is linear, a closed from solution is straightforward to determine. Indeed, the expected payoff with no externality would only be affected by an affine transformation. As the result from Baye et al. (1996) is invariant to affine transformations of expected utility, the mixed strategies are also invariant. Unfortunately, it is not possible to determine an analytic solution without providing a particular shape of the externality function h_i. This is the consequence of

[16]Remark that $\mathbb{E}\left(h_i(x_i, \boldsymbol{X}_{-i}) \mid \max_{j \neq i} X_j \leqslant x_i\right) = \begin{cases} \dfrac{1}{\Pi_{j \neq i} F_j(x_i)} \displaystyle\int_{[0, x_i]^n} h_i(x_i, \boldsymbol{x}_{-i}) \Pi_{j \neq i} dF_j(x_i) \text{ if } x_i > 0 \\ 0 \text{ otherwise} \end{cases}$ and $\mathbb{E}\left(h_i(x_i, \boldsymbol{X}_{-i}) \mid \exists j \neq i, X_j > x_i\right) = \begin{cases} \dfrac{1}{1 - \Pi_{j \neq i} F_j(x_i)} \displaystyle\int_{([0, x_i]^n)^c} h_i(x_i, \boldsymbol{x}_{-i}) \Pi_{j \neq i} dF_j(x_i) \text{ if } x_i > 0 \\ 0 \text{ otherwise} \end{cases}$ with $([0, x_i]^n)^c$ the complement of $[0, x_i]^n$.

the general mapping between x_i and $\boldsymbol{x}_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$. However we are able to show the existence of a bidding equilibrium in mixed strategies.

A substantial step to determine the existence of a Nash equilibrium was established by Reny (1999) via the better-reply security approach: any compact, bounded and measurable game, with strategy space subset of a Hausdorff linear topological space, possesses a Nash equilibrium if its mixed extension satisfies the better-reply security condition. Yet the payoff security of the mixed extension can be quite demanding to determine, and some recent papers establish conditions easier to check.[17] Prokopovych and Yannelis (2014) determined the uniform diagonal security condition and Monteiro and Page Jr (2007) established the uniform payoff security condition. Despite both guarantee the existence of a Nash equilibrium with mixed strategies for the standard all-pay auction, they do not apply to the all-pay auction with price externalities. For example consider $x_i = 0$ and suppose $x_j \neq 0$ for all $j \neq i$, then uniform payoff security condition is not satisfied.

Alternatively Simon and Zame (1990) established other existence results in discontinuous games that could also be applied. Following their results, we would need to determine a new game with an *endogenous* tie breaking rule to show the existence of a Nash equilibrium with mixed strategies. That would yet require a substantially more laborious work than the approach considered in this paper, which is the use of ordered fixed point theorem.

Moreover remark that the results Becker and Damianov (2006), using the Glicksberg and Fan fixed point theorem, cannot be applied here because of the heterogeneous values.

Proposition 1. *Given assumptions $A1 - A4$ a mixed strategy Nash equilibrium exists.*

Nonetheless, given that the solution is defined on a closed and convex set of continuous distribution functions subset of a Hausdorff linear topological space, we are able to show its existence by using the Schauder's fixed-point theorem. Remark that we can consider to apply neither the Knaster-Tarski's fixed-point theorem nor the Brouwer's fixed-point theorem because of the infinite-dimensional set of distribution functions. Other applications of the Schauder's fixed-point theorem are provided by Stokey et al. (1989) for overlapping-generations models, Fudenberg and Tirole (1991) for mixed strategy Nash equilibria with uncountable actions sets, Amir (1996), Curtat (1996) and Balbus et al. (2015) for Markov equilibrium in stochastic inter-generational games, Anderson et al. (1998) for logit equilibria in all-pay auctions and Fey (2008) for pure strategy Bayesian-Nash equilibria in rent-seeking contests.

Proof of Proposition 1. As in Anderson et al. (1998), our proof to apply the Schauder's fixed-point theorem consists in two steps[18]: to establish a continuous mapping and a set of functions uniformly bounded and equicontinuous.

In a preliminary step to determine a fixed point of this mapping, we first look at the indifference principle in our setting (leading to equation 2), and therefore establish a characterization

[17] See Reny (2020) for a discussion.

[18] However, we face an entire different problem than Anderson et al. (1998) as they show the existence of a logit equilibrium in all-pay auctions.

of the mapping (equation 3).

Preliminary Steps.

Let us consider two bidders, i and j. Notice that if the bidder i has the highest maximum bid $\bar{x}_i$, he obtains with probability 1 a payoff $v_i - \bar{x}_i + h_i(\bar{x}_i, x_j)$. Given Assumption A4, $\frac{\partial h_i}{\partial x_i}(x_i, x_j) \leqslant 1$, a decrease of $\bar{x}_i$ to the maximum of her rival, denoted $\bar{x}_j$, will increase her payoff to $v_i - \bar{x}_j + h_i(\bar{x}_j, x_j)$ without altering her probability of winning. Following the standard reasoning for multiple bidders developed by Baye et al. (1996), all active bidders must have the same maximum bid $\bar{x}$.[19]

Preliminary Step 1.

We now investigate the indifference principle, stating that if all bidder i's opponents play a mixed strategy, bidder i is indifferent among all pure strategies in the possible best response set, in other words that the bidder's i expected utility is constant for any pure strategy profile of the other bidders with positive probability. Let us denote $G_i := \prod_{j=1, j\neq i}^{n} F_j$ and the vector $\boldsymbol{y}_{-i} = (y_1, ..., y_{i-1}, y_{i+1}, ..., y_n)$. Given Assumption A2 and the use of the Stieljes integral, the indifference principle leads to the derivative:

$$G_i'(x) = \frac{1}{v_i} - \frac{1}{v_i} \int_{[0,\bar{x}]^{n-1}} \frac{\partial h_i}{\partial x}(x, \boldsymbol{y}_{-i}) \Pi_{j\neq i} dF_j(y_j) \text{ for all } i = 1, ..., n. \tag{2}$$

Preliminary Step 2.

In the following, we show that equation (2) is well defined. Notice that the right side of equation (2) involves G_i to be a continuous distribution function. We denote $\boldsymbol{G}(x) = (G_1(x), ..., G_n(x))$ the vector of mixed strategies and $\mathcal{D}_i = \{G_i \in \mathcal{C}([0,\bar{x}]), ||G_i|| \leqslant 1\}$ the set of distribution functions continuous on $[0,\bar{x}]$, with $||.||$ the supremum norm, and $\mathcal{C}([0,\bar{x}])$ the set of continuous functions on $[0,\bar{x}]$. It follows the set $\boldsymbol{\mathcal{D}} \equiv \mathcal{D}_1 \times ... \times \mathcal{D}_n$ with the norm $||\boldsymbol{G}||_\infty = \max_{i=1,...,n} ||G_i||$. Therefore let us consider $T : \boldsymbol{\mathcal{D}} \longrightarrow \boldsymbol{\mathcal{D}}$ to be the mapping $\boldsymbol{G}(x) \longmapsto \boldsymbol{TG}(x)$, with $\boldsymbol{TG}(x) = (TG_1(x), ..., TG_n(x))$. Then T maps every element G_i from $\mathcal{D}_i$ to $\mathcal{D}_i$, which given the integration of equation (2), is a function characterized by

$$TG_i(x) \equiv \lambda_i x - \lambda_i \int_{[0,\bar{x}]^{n-1}} h_i(x, \boldsymbol{y}_{-i}) \Pi_{j\neq i} dF_j(y_j) \text{ for all } i = 1, ..., n. \tag{3}$$

with $\lambda_i = \frac{1}{v_i}$.

[19] The proof for n bidders and all possible rankings of valuations is similar to the one without externality in Baye et al. (1996).

Remark that the set $\boldsymbol{\mathcal{D}}$, which includes all the continuous distribution functions, is closed and convex but also infinite-dimensional, subset of a Hausdorff linear topological space.[20] The Schauder's fixed-point theorem is then required to prove that $\boldsymbol{G}(x)$ is a fixed-point of the operator T defined by equation (3). It provides a sufficient condition that the mixed strategies are a fixed point of the operator T, and then there is no profitable deviations outside the support $[0, \bar{x}]$.

Main Steps.

Theorem 1 (Schauder, 1930)**.** *If $\boldsymbol{\mathcal{D}}$ is a closed convex subset of a normed space and $\boldsymbol{\mathcal{E}}$ a relatively compact subset of $\boldsymbol{\mathcal{D}}$, then every continuous mapping of $\boldsymbol{\mathcal{D}}$ to $\boldsymbol{\mathcal{E}}$ has a fixed-point.*

Therefore, we show in the following that $\boldsymbol{\mathcal{E}} \equiv \{\boldsymbol{TG} \mid \boldsymbol{G} \in \boldsymbol{\mathcal{D}}\}$ is relatively compact and that T is a continuous operator from $\boldsymbol{\mathcal{D}}$ to $\boldsymbol{\mathcal{E}}$.

Step 1. *$\boldsymbol{\mathcal{E}}$ is relatively compact.*

We use the Arzelà-Ascoli's theorem to characterize the relative compactness in the space of continuous functions $\mathcal{C}([0, \bar{x}])$.

Theorem 2 (Arzelà-Ascoli, 1895)**.** *A set of functions in $\mathcal{C}([0, \bar{x}])$, with the supremum norm, is relatively compact if and only if it is uniformly bounded and equicontinuous on $[0, \bar{x}]$.*

Thus, to establish that $\boldsymbol{\mathcal{E}} \equiv \{\boldsymbol{TG} \mid \boldsymbol{G} \in \boldsymbol{\mathcal{D}}\}$ is relatively compact, we prove that $\boldsymbol{\mathcal{E}}$ is uniformly bounded and equicontinuous on $[0, \bar{x}]$.

First let us show that $\boldsymbol{\mathcal{E}}$ is uniformly bounded. Assumption $A4$ implies that

$$\int_{[0,\bar{x}]^{n-1}} \frac{\partial h_i}{\partial x}(x, \boldsymbol{y}_{-i}) \Pi_{j \neq i} dF_j(y_j) \leqslant 1.$$

$TG_i(x)$ is thus increasing and $|TG_i(x)| \leqslant TG_i(\bar{x}) = 1$, for all $x \in [0, \bar{x}]$, $G_i \in \mathcal{D}_i$, $i = 1, ..., n$. Therefore, $\boldsymbol{TG}$ is uniformly bounded for all $\boldsymbol{G} \in \boldsymbol{\mathcal{D}}$.

Second, let us now prove that $\boldsymbol{TG}$ is equicontinuous $\forall \boldsymbol{G} \in \boldsymbol{\mathcal{D}}$: $\forall \varepsilon, \exists \eta : |TG_i(x_1) - TG_i(x_2)| < \varepsilon$ when $|x_1 - x_2| < \eta$, $\forall G_i \in \mathcal{D}_i$ and $i = 1, ..., n$. To show it, notice that the function h_i is continuous and bounded on the compact $[0, \bar{x}]$. We can then compute,

$$\begin{aligned}
|TG_i(x_1) - TG_i(x_2)| &= \left| \lambda_i (x_1 - x_2) - \lambda_i \int_{[0,\bar{x}]^{n-1}} [h_i(x_1, \boldsymbol{y}_{-i}) - h_i(x_2, \boldsymbol{y}_{-i})] \Pi_{j \neq i} dF_j(y_j) \right| \\
&\leqslant \lambda_i \left[|x_1 - x_2| + \left| \int_{[0,\bar{x}]^{n-1}} [h_i(x_1, \boldsymbol{y}_{-i}) - h_i(x_2, \boldsymbol{y}_{-i})] \Pi_{j \neq i} dF_j(y_j) \right| \right] \\
&\leqslant \lambda_i |x_1 - x_2| \left[1 + \frac{|\sup_{\boldsymbol{y}_{-i} \in [0,\bar{x}]^{n-1}} [h_i(x_1, \boldsymbol{y}_{-i}) - h_i(x_2, \boldsymbol{y}_{-i})]|}{|x_1 - x_2|} \right] \\
&< \lambda_i \eta \left[1 + \frac{|\sup_{\boldsymbol{y}_{-i} \in [0,\bar{x}]^{n-1}} [h_i(x_1, \boldsymbol{y}_{-i}) - h_i(x_2, \boldsymbol{y}_{-i})]|}{|x_1 - x_2|} \right].
\end{aligned}$$

[20] Equation (2) guarantees the distribution functions G_i are continuous.

Thus, $|TG_i(x_1)-TG_i(x_2)| < \varepsilon$ for $\eta = \varepsilon \min_{i=1,...,n} \frac{|x_1-x_2|}{\lambda_i(|x_1-x_2|+\kappa_i)}$ for all $G_i \in \mathcal{D}_i$ and $i = 1, ..., n$, with $\kappa_i \equiv |\sup_{\boldsymbol{y}_{-i}\in[0,\bar{x}]^{n-1}}[h_i(x_1,\boldsymbol{y}_{-i}) - h_i(x_2,\boldsymbol{y}_{-i})]|$.

Step 2. *T is a continuous mapping from $\boldsymbol{\mathcal{D}}$ to $\boldsymbol{\mathcal{E}}$.*

To establish T is a continuous mapping, we define $\hat{G}_i(x) = \Pi_{j\neq i}\hat{F}_j(x)$ and $\tilde{G}_i(x) = \Pi_{j\neq i}\tilde{F}_j(x)$ such as $\hat{G}_i(x) = \tilde{G}_i(x) + k_i(x)$ with $|k_i(x)| < \eta$ for all $x \in [0,\bar{x}]$, $i = 1, ..., n$, and show that for all $\hat{\boldsymbol{G}}, \tilde{\boldsymbol{G}} \in \boldsymbol{\mathcal{D}}$ and for all $\varepsilon > 0$, there is a $\eta > 0$ such that $||\boldsymbol{T}\hat{\boldsymbol{G}}(x) - \boldsymbol{T}\tilde{\boldsymbol{G}}(x)||_\infty < \varepsilon$ when $||\hat{\boldsymbol{G}} - \tilde{\boldsymbol{G}}||_\infty < \eta$. We can compute,

$$
\begin{aligned}
|T\hat{G}_i(x) - T\tilde{G}_i(x)| &= \left| -\lambda_i \left(\int_{[0,\bar{x}]^{n-1}} h_i(x,\boldsymbol{y}_{-i})\Pi_{j\neq i} d\hat{F}_j(y_j) - \int_{[0,\bar{x}]^{n-1}} h_i(x,\boldsymbol{y}_{-i})\Pi_{j\neq i} d\tilde{F}_j(y_j) \right) \right| \\
&\leqslant \lambda_i \sup_{\boldsymbol{y}_{-i}\in[0,\bar{x}]^{n-1}} h_i(\bar{x},\boldsymbol{y}_{-i}) \left| \int_{[0,\bar{x}]^{n-1}} \Pi_{j\neq i} d\hat{F}_j(y_j) - \int_{[0,\bar{x}]^{n-1}} \Pi_{j\neq i} d\tilde{F}_j(y_j) \right| \\
&= \lambda_i \sup_{\boldsymbol{y}_{-i}\in[0,\bar{x}]^{n-1}} h_i(\bar{x},\boldsymbol{y}_{-i}) |k_i(\bar{x})| \\
&< \lambda_i \sup_{\boldsymbol{y}_{-i}\in[0,\bar{x}]^{n-1}} h_i(\bar{x},\boldsymbol{y}_{-i})\eta.
\end{aligned}
$$

As h_i is a continuous function in all arguments, bounded by $\sup_{\boldsymbol{y}_{-i}\in[0,\bar{x}]^{n-1}} h_i(\bar{x},\boldsymbol{y}_{-i})$, the second line follows. The transition from the second to the third line is from the independence of the density functions and $\hat{G}_i(x) - \tilde{G}_i(x) = k_i(x)$. Therefore, $||\boldsymbol{T}\hat{\boldsymbol{G}}(x) - \boldsymbol{T}\tilde{\boldsymbol{G}}(x)||_\infty$ is inferior to $\varepsilon > 0$ when $\eta = \min_{i=1,...,n} \frac{\varepsilon}{\lambda_i \sup_{\boldsymbol{y}_{-i}\in[0,\bar{x}]^{n-1}} h_i(\bar{x},\boldsymbol{y}_{-i}))}$ for all $x \in [0,\bar{x}]$.

■

4 Concluding Remarks

We show the existence of a mixed strategy Nash equilibrium for all-pay auctions with general price externalities. The proof relies on the Schauder's fixed-point theorem, which does not require a finite-dimensional set of the distribution functions. Our result can be useful for many economic applications such as charity mechanisms and contests in which relative performance matters.

Unfortunately, there is no closed form solution to this problem without providing a specific form of the externality functions. A numerical approach combined with a lab experiment would be useful to determine how non-linearity can affect the revenue performance of the all-pay auction. Uniqueness is also an important investigation for future research. The two-bidder case leads to a first-order condition which can be identified as a Fredholm equation. Kanwal (1971) provides a sufficient condition for uniqueness, $\sup_{x\in[0,\bar{x}]} \int_0^{\bar{x}} \left|\frac{\partial h_i}{\partial x_i}(x_i,x_j)\right| dx_j < 1$, which is quite restrictive and seems unrelevant for an economic analysis.

To sum up, natural follow-up questions concern the extent of uniqueness and numerical approach to determine the properties of the equilibrium. These could have many interests for applications in fundraising mechanisms and race contests.

This paper also raised an important issue about the use of the all-pay auction for charity purpose. While theoretical results established it is an optimal mechanism (Goeree et al., 2005, Engers and McManus, 2007), practice on the field conclude the opposite (Carpenter et al., 2008, Onderstal et al., 2013). One possible explanation is a misleading assumption about linear bidders' preferences for charity. Our contribution rises a new question about optimal mechanisms at raising money for charity with non-linear externalities. This would also require an analysis about the relative performance of the all-pay auction compared to other mechanisms widely used in practice such as lotteries and voluntary contributions.

5 Appendix

We proposes in this section a sketch of the proof of the no bidding equilibrium in pure strategies in all-pay auctions. Only the two-bidder case is investigated to make the argument easier to follow.

Let us assume that $x_i \geqslant x_j$, then two cases may arise. First if bidder j can overbid, her profitable deviation is $x_i + \varepsilon$ for $\varepsilon > 0$, such that $v_j - (x_i + \varepsilon) + h_j(x_i, x_i + \varepsilon) \geqslant -x_j + h_j(x_i, x_j)$. In this case, $x_i \geqslant x_j$ is not possible. Second if j cannot overbid, her profitable deviation is to offer zero since $h_j(0, x_i) > -x_j + h_j(x_j, x_i)$ given $\frac{\partial h_j}{\partial x_j}(x_j, x_i) < 1$. Therefore, i's profitable deviation is to offer $\varepsilon > 0$. As a result, this is unstable and there is no pure strategy Nash equilibrium.

There are also many situations in our general case described by assumption A4, $\frac{\partial h_i}{\partial x_i}(x_i, \boldsymbol{x}_{-i}) \leqslant 1$ for all $i = 1, ..., n$, in which there is no Nash equilibrium in pure strategies. To make the statement more straightforward to follow, we focus again on the two-bidder case.

Let suppose both bidders maximize $v_i - x_i + h_i(x_i, x_j)$ for $i = 1, 2, i \neq j$. Then they choose $(\tilde{x}_1, \tilde{x}_2)$ such that

$$\frac{\partial h_1}{\partial x_1}(\tilde{x}_1, \tilde{x}_2) = \frac{\partial h_2}{\partial x_2}(\tilde{x}_2, \tilde{x}_1) = 1.$$

If both bidders benefit from the same externality functions, $h_1 = h_2 \equiv h$, they bid $\tilde{x}_1 = \tilde{x}_2 = \tilde{x}$ and get a payoff $\frac{v_i}{2} - \tilde{x} + h(\tilde{x}, \tilde{x})$ for $i = 1, 2$. Let us now consider $v_1 > v_2$ such that bidders do not have the same maximum bid. In this case, bidder i is always better off by overbidding $x_i = \tilde{x} + \varepsilon$ for $\varepsilon > 0$ and $\tilde{x}_1 = \tilde{x}_2 = \tilde{x}$ cannot be an equilibrium.[21] Using the Taylor's theorem at the point $\tilde{x}$ such that $h(\tilde{x} + \varepsilon, \tilde{x}) = h(\tilde{x}, \tilde{x}) + \varepsilon \frac{\partial h}{\partial x_i}(\tilde{x}, \tilde{x}) + o(\varepsilon)$ with $\lim_{\varepsilon \to 0} o(\varepsilon) = 0$, a bid $\tilde{x} + \varepsilon$ leads to the payoff $v_i - \tilde{x} - \varepsilon + h(\tilde{x}, \tilde{x}) + \varepsilon \frac{\partial h}{\partial x_i}(\tilde{x}, \tilde{x}) + o(\varepsilon)$. Therefore, as $\frac{\partial h}{\partial x_i}(\tilde{x}, \tilde{x}) = 1$, overbidding $\tilde{x} + \varepsilon$ provides a higher payoff to bidder i. The best reply of bidder j is either to overbid $\tilde{z}$, if $v_j - \tilde{z} + h(\tilde{z}, \tilde{x} + \varepsilon) > -\tilde{x} + h(\tilde{x}, \tilde{x} + \varepsilon)$, or to underbid $\tilde{y}$.

[21]If $v_1 = v_2 = v$, both bidders have the same maximum bid $\bar{x}$. The unique possible symmetric bidding equilibrium in pure strategies, $\tilde{x}_1 = \tilde{x}_2 = \tilde{x}$, is the highest $\tilde{x}$ such that $\tilde{x} \leqslant \bar{x}$ and $\frac{\partial h}{\partial x_i}(\bar{x}, \bar{x}) = 1$.

Excluding the particular case $\frac{\partial h}{\partial x_j}(x_j, \tilde{x} + \varepsilon) = 1$ for all $x_j < \tilde{x}$, there is at least a value $\tilde{y}$ such that $\frac{\partial h}{\partial x_j}(\tilde{y}, \tilde{x} + \varepsilon) < 1$ is satisfied. Then bidder j underbids the smallest possible $\tilde{y}$.[22]. Therefore, ruling out the particular case $\frac{\partial h}{\partial x_i}(x_i, \tilde{y}) = 1$ for all $x_i \in (\tilde{y}, \tilde{x} + \varepsilon]$, there is at least a value $\tilde{y} + \kappa$ with $\kappa > 0$ such that bidder i underbids. Bidder j will then deviate by proposing again $\tilde{x}$ if v_2 is sufficiently high. Therefore, this is unstable and there is no Nash equilibrium in pure strategies. This result of the potential non-existence of bidding equilibrium in pure strategies can be extended to heterogeneous externality functions with a similar reasoning.

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[22]Indeed, $-\tilde{y} + h(\tilde{y}, \tilde{x} + \varepsilon) > -\tilde{x} + h(\tilde{x}, \tilde{x} + \varepsilon)$.

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